# FROBENIUS MORPHISMS AND REPRESENTATIONS OF ALGEBRAS

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Abstract. By introducing Frobenius morphisms $F$ on algebras $A$ and their modules over the algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ of $q$ elements, we establish a relation between the representation theory of $A$ over $\overline{\mathbb{F}}_q$ and that of the $F$-fixed point algebra $A^F$ over $\mathbb{F}_q$. More precisely, we prove that the category $\mathbf{mod}$-$A^F$ of finite dimensional $A^F$-modules is equivalent to the subcategory of finite dimensional $F$-stable $A$-modules, and, when $A$ is finite dimensional, we establish a bijection between the isoclasses of indecomposable $A^F$-modules and the $F$-orbits of the isoclasses of indecomposable $A$-modules. Applying the theory to representations of quivers with automorphisms, we show that representations of a modulated quiver (or a species) over $\mathbb{F}_q$ can be interpreted as $F$-stable representations of a corresponding quiver over $\overline{\mathbb{F}}_q$. We further prove that every finite dimensional hereditary algebra over $\mathbb{F}_q$ is Morita equivalent to some $A^F$, where $A$ is the path algebra of a quiver $Q$ over $\overline{\mathbb{F}}_q$ and $F$ is induced from a certain automorphism of $Q$. A close relation between the Auslander-Reiten theories for $A$ and $A^F$ is established. In particular, we prove that the Auslander-Reiten (modulated) quiver of $A^F$ is obtained by "folding" the Auslander-Reiten quiver of $A$. Finally, by taking Frobenius fixed points, we are able to count the number of indecomposable representations of a modulated quiver with a given dimension vector and to establish part of Kac's theorem for all finite dimensional hereditary algebras over a finite field.

CONTENTS



## 1. Introduction

In his work [11], Gabriel introduced the idea of quiver representations and discovered a remarkable connections between the indecomposable representations of (simply-laced) Dynkin quivers and the positive roots of the corresponding finite dimensional simple Lie algebras. The theory of quiver representations not only may be viewed as a new language for the whole range of problems in linear algebra, but also provided a platform (over an algebraically closed field) for bringing new ideas and techniques from algebraic geometry and Lie theory into the subject. There are two major fundamental developments after Gabriel's work. In order to complete Gabriel's classification to include all Dynkin graphs, Dlab and Ringel [8] studied representations of a modulated quiver (or a species) and proved that a modulated quiver admits only finitely many indecomposable

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representations if and only if it is of Dynkin type. In the subsequent works [10, 25, 9], representations of both quivers and modulated quivers of "tame" representation type are classified. On the other hand, by employing the methods of invariant theory, Kac [19, 20] was able to establish a connection between the indecomposable representations of *any* finite quiver and the positive roots of the corresponding Kac-Moody algebra with a symmetric generalized Cartan matrix.

These fundamental works provide two major approaches in the representation theory of algebras. The quiver approach is usually used to study representations of algebras over an algebraically closed field in which methods in algebraic geometry and invariant theory can be applied. However, in the context of Lie algebras, it only deals with the case of symmetric (generalized) Cartan matrices. The modulated quiver approach, though a bit artificial, is purely algebraic, and suitable for an *arbitrary* ground field. If the underlying valued quiver of a modulated quiver is not constantly valued, then the corresponding symmetrizable Cartan matrix is not symmetric. Thus, this approach covers the case of all symmetrizable Cartan matrices.

Another important achievement in representation theory of algebras is the discovery of Auslander-Reiten sequences by Auslander and Reiten in 1970's. Such sequences reflect the additional structures imposed on the category of finite dimensional modules over an algebra by the existence of kernels and cokernels. The Auslande-Reiten theory soon became a fundamental tool in the study of representations of algebras.

It is well-known in the Lie theory that a non-symmetric Cartan matrix can be obtained by "folding" the graph of a symmetric Cartan matrix via a graph automorphism. Such an idea has been used in [27, 23, 24, 26, 18] to study representations of quivers with an automorphism. In this paper we shall extend this idea, combining with the idea of Frobenius morphisms in the theory of algebraic groups, to build a direct bridge between the quiver and modulated quiver approaches. By introducing Frobenius morphisms of algebras $A$ defined over $\overline{\mathbb{F}}_q$, we shall prove that representations of the fixed-point algebra $A^F$ are obtained by taking fixed points of $F$-stable representations of $A$. In particular, if $A$ is the path algebra of a quiver $Q$ which admits an admissible automorphism $\sigma$ and $F$ is the Frobenius morphism on $A$ induced from $\sigma$, then $A^F$ is isomorphic to the tensor algebra of the modulated quiver obtained by folding $Q$ and $A$ through $\sigma$ and $F$, respectively. Thus, the representation theory of $\mathbb{F}_q$-modulated quiver can be realized as that of an ordinary quiver $Q$ by simply taking fixed-points of $F$-stable representations of $Q$ over $\overline{\mathbb{F}}_q$. Further, we establish a relation between the Auslander-Reiten theories of $A$ and its fixed point algebra $A^F$.

We organize the paper as follows. Section 2 is a brief introduction on $\mathbb{F}_q$-structures of vector spaces. In §3 we consider algebras $A$ with Frobenius morphisms $F$ and define $F$-stable $A$-modules. Then we show that the category of $F$-stable $A$-modules is isomorphic to the category of modules over the fixed point algebra $A^F$. In §4 we define the (Frobenius) twist of an $A$-module and introduce the notion of $F$-periodic $A$-modules. As a result, we prove in §5 that each indecomposable $A^F$-module can be obtained by "folding" $F$-periodic $A$-modules. In particular, if $A$ is finite dimensional, there is a bijection between indecomposable $A^F$-modules and $F$-orbits of the indecomposable $A$-modules. As a first application to representations of quivers with automorphisms, it is shown in §6 that the representation theory of a modulated quiver (or a species) over $\mathbb{F}_q$ is completely determined by the representation theory of the corresponding quiver over $\overline{\mathbb{F}}_q$. We further prove that every finite dimensional hereditary algebra over $\mathbb{F}_q$ is Morita equivalent to some $A^F$, where $A$ is the path algebra of a quiver $Q$ over $\overline{\mathbb{F}}_q$ and $F$ is the Frobenius morphism induced from an automorphism of $Q$. In §7 we establish a relation between almost split sequences of $A$ and $A^F$. Section 8 is devoted to proving that the Auslander-Reiten (modulated) quiver of $A^F$ is obtained by "folding" the Auslander-Reiten quiver of $A$. In §9 and §10, we present formulae of the number of indecomposable $F$-stable representations of an ad-quiver with a fixed dimension vector and prove

part of Kac's theorem for all finite dimensional hereditary algebras over a finite field. Thus, we reobtain and generalize some results in [14, 4, 17].

In [5, 6], a strong monomial basis property for quantized enveloping algebras of simply-laced Dynkin or cyclic type was discovered. The present work was motivated by seeking a similar property in the non-simply-laced case. Though methods such as representation varieties and generic extensions employed in [5, 6] are no longer valid when working over a finite field, we want a new approach that carries them over. In a forthcoming paper, we shall apply the theory developed in the paper to complete our tasks on the strong monomial basis property for *all* quantized enveloping algebras of finite type.

*Throughout, let $q$ be a prime power, $\mathbb{F}_q$ the finite field of $q$ elements and $k$ the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. For any $r \geqslant 1$, let $\mathbb{F}_{q^r}$ denote the unique extension field of $\mathbb{F}_q$ of degree $r$ contained in $k = \overline{\mathbb{F}}_q$. All modules considered are left modules of finite dimension over the base field. If $M$ is a module, $[M]$ denotes the class of modules isomorphic to $M$, i.e., the isoclass of $M$. For any field $\mathtt{k}$, the notation $\mathtt{k}^{m\times n}$ denotes the set of all $m \times n$ matrices over $\mathtt{k}$.*

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## 2. $\mathbb{F}_q$-structures on vector spaces

An $\mathbb{F}_q$-*structure* on a vector space $V$ over $k$ is an $\mathbb{F}_q$-subspace $V_0$ of $V$ viewed as a space over $\mathbb{F}_q$ such that the canonical homomorphism $V_0 \otimes_{\mathbb{F}_q} k \to V$ is an isomorphism. We shall always identify $V$ with $V_0 \otimes k$ in the sequel.

**Lemma 2.1.** *A $k$-space $V$ has an $\mathbb{F}_q$-structure $V_0$ if and only if*

$$V_0 = V^F := \{v \in V \mid F(v) = v\}$$

*for some $\mathbb{F}_q$-linear isomorphism $F : V \to V$ satisfying*

(a) *$F(\lambda v) = \lambda^q F(v)$ for all $v \in V$ and $\lambda \in k$;*
(b) *for any $v \in V$, $F^n(v) = v$ for some $n > 0$.*

*Proof.* If $V$ has an $\mathbb{F}_q$-structure $V_0$, then we have $V = V_0 \otimes_{\mathbb{F}_q} k$ and define $F : V \to V$ by sending $v \otimes a$ to $v \otimes a^q$. Clearly, $V_0 = V^F$ and $F$ satisfies the conditions (a) and (b). The proof for the converse is given in [7, 3.5]. □

The map $F$ is called a *Frobenius map*. By the lemma, we see that an $\mathbb{F}_q$-structure on $V$ is equivalent to the existence of a Frobenius map. However, different Frobenius morphisms may define the same $\mathbb{F}_q$-structure. From the proof we see that if $F : V \to V$ is a Frobenius map, then there is a basis $\{v_i\}$ of $V$ such that $F(\sum_i \lambda_i v_i) = \sum_i \lambda_i^q v_i$.

**Corollary 2.2.** *If $F$ and $F'$ are Frobenius maps on a* finite dimensional *space $V$, then $F' \circ F^{-1}$ is linear on $V$ and there is a positive integer $n$ such that $F^n = F'^n$.*

*Proof.* Since $V = V^F \otimes_{\mathbb{F}_q} k = V^{F'} \otimes_{\mathbb{F}_q} k$ and $V$ is finite dimensional, there is a positive integer $n$ such that

$$V(\mathbb{F}_{q^n}) := V^F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n} = V^{F'} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}.$$

Choose bases $\{v_i\}$ and $\{w_i\}$ for $V^F$ and $V^{F'}$, respectively, and write $w_j = \sum_j x_{ij} v_i$ with $x_{ij} \in \mathbb{F}_{q^n}$. Now, one checks easily that $F^n = F'^n$ on $V(\mathbb{F}_{q^n})$, and hence, on $V$. □

Let $\mathcal{V}_{k,\mathbb{F}_q}$ be the category whose objects are vector spaces over $k$ with *fixed* $\mathbb{F}_q$-structures and whose morphisms are linear maps $f : V \to W$ defined over $\mathbb{F}_q$, namely, $f(V_0) \subseteq W_0$. Clearly, if $F_V$ and $F_W$ are the Frobenius maps on $V$ and $W$ defining $V_0$ and $W_0$, respectively, then $f$ is defined over $\mathbb{F}_q$ if and only if $F_W \circ f = f \circ F_V$. In particular, if $F$ and $F'$ are two Frobenius maps on $V$, then any $\mathbb{F}_q$-linear isomorphism from $V^F$ to $V^{F'}$ induces a $k$-linear isomorphism $\theta$ on $V$ such that $\theta \circ F = F' \circ \theta$. In other words, up to isomorphism in $\mathcal{V}_{k,\mathbb{F}_q}$, the $\mathbb{F}_q$-structure on $V$ is unique. Note that $\mathcal{V}_{k,\mathbb{F}_q}$ is an abelian category.

We shall represent the fixed $\mathbb{F}_q$-structure on a $k$-space $V$ by a Frobenius map $F_V$. Thus, the objects $V$ in $\mathcal{V}_{k,\mathbb{F}_q}$ are, sometimes, written as $(V, F_V)$, and

$$\mathrm{Hom}_{\mathcal{V}_{k,\mathbb{F}_q}}(V,W) = \{f \in \mathrm{Hom}_k(V,W) \mid F_W \circ f = f \circ F_V\}.$$

For $(V, F_V), (W, F_W) \in \mathrm{Ob}(\mathcal{V}_{k,\mathbb{F}_q})$, there is an induced $\mathbb{F}_q$-linear map

$$F_{(V,W)} : \mathrm{Hom}_k(V,W) \to \mathrm{Hom}_k(V,W); f \mapsto F_{(V,W)}(f) = F_W \circ f \circ F_V^{-1}.$$

Note that, when both $V$ and $W$ are *infinite dimensional*, $F_{(V,W)}$ is not necessarily a Frobenius map on $\mathrm{Hom}_k(V,W)$ in the sense of Lemma 2.1. However, we have the following.

**Lemma 2.3.** *For $(V_1, F_1), (V_2, F_2) \in \mathrm{Ob}(\mathcal{V}_{k,\mathbb{F}_q})$, let $F = F_{(V_1,V_2)}$ be defined as above.*

*(a) We have an $\mathbb{F}_q$-space isomorphism*

$$\mathrm{Hom}_{\mathbb{F}_q}(V_1^{F_1}, V_2^{F_2}) \cong \mathrm{Hom}_{\mathcal{V}_{k,\mathbb{F}_q}}(V_1, V_2) = \mathrm{Hom}_k(V_1, V_2)^F.$$

*(b) Let $\mathrm{hom}_k(V_1, V_2) = \mathrm{Hom}_k(V_1, V_2)^F \otimes k$. Then, $F$ is a Frobenius map on $\mathrm{hom}_k(V_1, V_2)$ and, if $V_i$ are finite dimensional, then $\mathrm{hom}_k(V_1, V_2) = \mathrm{Hom}_k(V_1, V_2)$.*

*Proof.* The isomorphism in (a) is defined by sending $f$ to $f\otimes 1$ with inverse defined by restriction. To see (b), it is clear that $F$ is an $\mathbb{F}_q$-linear map satisfying 2.1(a) and $F$ satisfies 2.1(b) on $\mathrm{hom}_k(V_1, V_2)$. Finally, the last assertion follows from a dimension comparison. $\square$

**Remarks 2.4.** (1) Every Frobenius map on a finite dimensional $k$-space $V$ induces the Frobenius map $F = F_{(V,V)}$ on the algebra $\mathrm{End}_k(V)$ of all linear transformations on $V$. Clearly, $F$ is an $\mathbb{F}_q$-algebra automorphism and induces an $\mathbb{F}_q$-algebra isomorphism $\mathrm{End}_k(V)^F \cong \mathrm{End}_{\mathbb{F}_q}(V^F)$.

(2) The restriction of $F$ on $\mathrm{End}_k(V)$ to the general linear group $GL(V)$ induces a Frobenius morphism on the connected algebraic group $GL(V)$ to which the following Lang-Steinberg theorem apply.

**Theorem 2.5.** (**Lang-Steinberg**) *Let $G$ be a connected affine algebraic group and let $F$ be a surjective endomorphism of $G$ with a finite number of fixed points. Then the map $\mathcal{L} : g \mapsto g^{-1}F(g)$ from $G$ to itself is surjective.*

## 3. Algebras with Frobenius morphisms

Let $A$ be a $k$-algebra with identity 1. We do not assume generally that $A$ is finite dimensional. A map $F_A : A \to A$ is called a *Frobenius morphism* on $A$ if it is a Frobenius map on the $k$-space $A$ and it is also an $\mathbb{F}_q$-algebra isomorphism sending 1 to 1. For example, for a finite dimensional $k$-space $V$, $A = \mathrm{End}_k(V)$ admits a Frobenius morphism $F_{(V,V)}$ induced from a Frobenius map $F_V$ on $V$ (see 2.4).

Given a Frobenius morphism $F_A$ on $A$, let

$$A^F := A^{F_A} = \{a \in A \mid F_A(a) = a\}$$

be the set of $F_A$-fixed points. Then $A^F$ is an $\mathbb{F}_q$-subalgebra of $A$, and $A = A^F \otimes k$. The Frobenius morphism $F_A$ induces an algebra isomorphism on $A^F$ and $F_A(a \otimes \lambda) = F_A(a) \otimes \lambda^q$ for all $a \in A^F, \lambda \in k$.

Let $M$ be a finite dimensional $A$-module and let $\pi = \pi_M : A \to \mathrm{End}_k(M)$ be the corresponding representation. We say that $M$ is *$F_A$-stable* (or simply $F$-stable) if there is an $\mathbb{F}_q$-structure $M_0$ of $M$ such that $\pi$ induces a representation $\pi_0 : A^F \to \mathrm{End}_{\mathbb{F}_q}(M_0)$ of $A^F$. Clearly, by Lemma 2.1, $M$ is $F$-stable if and only if there is a Frobenius map $F_M : M \to M$ such that

$$F_M(am) = F_A(a)F_M(m), \quad \text{for all } a \in A, m \in M. \tag{3.0.1}$$

In terms of the corresponding representation $\pi$, the $F$-stability of $M$ simply means that there is a Frobenius map $F_M$ on $M$ such that $\pi \circ F_A = F_{(M,M)} \circ \pi$. *In the sequel, we shall fix such an $\mathbb{F}_q$-structure $M_0$ for an $F$-stable module and represent it by a Frobenius map $F_M$.* Thus, if $M$ is an $F$-stable $A$-module with respect to $F_M$, then $M = M^F \otimes_{\mathbb{F}_q} k$, $F_M(m \otimes \lambda) = m \otimes \lambda^q$ for all $m \in M^F$, $\lambda \in k$, and $M^F$ is an $A^F$-module. Here, again for notational simplicity, we write $M^F$ for $M^{F_M}$.[1] We shall also use, sometimes, the notation $(M, F_M)$ for an $F$-stable module $M$.

**Lemma 3.1.** *Let $(M_1, F_1)$ and $(M_2, F_2)$ be two $F$-stable $A$-modules. Then $M_1^{F_1} \cong M_2^{F_2}$ as $A^F$-modules if and only if $M_1 \cong M_2$ as $A$-modules. In particular, if $M_1 = M_2 = M$, then $M^{F_1} \cong M^{F_2}$ as $A^F$-modules.*

*Proof.* Since $M_i = M_i^{F_i} \otimes_{\mathbb{F}_q} k$ and $A = A^F \otimes_{\mathbb{F}_q} k$, the lemma follows directly from Noether-Deuring Theorem (see for example [3, p.139]). $\square$

This result shows that, up to isomorphism, it doesn't matter which Frobenius maps (or $\mathbb{F}_q$-structures) on $M$ we shall work with when considering $F$-stable modules.

Let $\mathbf{mod}^F\text{-}A$ denote the category whose objects are $F$-stable modules $M = (M, F_M)$. The morphisms from $(M_1, F_1)$ to $(M_2, F_2)$ are given by homomorphisms $\theta \in \mathrm{Hom}_A(M_1, M_2)$ such that $\theta \circ F_1 = F_2 \circ \theta$, that is, $A$-module homomorphisms compatible with their $\mathbb{F}_q$-structures. Clearly, $\mathbf{mod}^F\text{-}A$ is a subcategory of $\mathbf{mod}\text{-}A$. It is also easy to see that $\mathbf{mod}^F\text{-}A$ is an abelian $\mathbb{F}_q$-category.

The next result allows us to embed a module category defined over a finite field into a category defined over the algebraic closure of the finite field.

**Theorem 3.2.** *The abelian category $\mathbf{mod}^F\text{-}A$ is equivalent to the category $\mathbf{mod}\text{-}A^F$ of finite dimensional $A^F$-modules.*

*Proof.* Let $M = (M, F_M)$ be an object in $\mathbf{mod}^F\text{-}A$. We define $\Phi(M) := M^F$, which is an $A^F$-module. Now let $\theta : (M_1, F_2) \to (M_2, F_2)$ be a morphism in $\mathbf{mod}^F\text{-}A$. Since $\theta \circ F_1 = F_2 \circ \theta$, $\theta$ induces a map $\Phi(\theta) : M_1^{F_1} \to M_2^{F_2}$ which is obviously an $A^F$-module homomorphism. This gives a functor $\Phi : \mathbf{mod}^F\text{-}A \to \mathbf{mod}\text{-}A^F$.

Conversely, for each $A^F$-module $X$, we set $\Psi(X) = X \otimes_{\mathbb{F}_q} k$ and define a Frobenius map

$$F_{\Psi(X)} : \Psi(X) \longrightarrow \Psi(X);\ x \otimes \lambda \longmapsto x \otimes \lambda^q.$$

By defining $(a \otimes \lambda)(x \otimes \mu) = ax \otimes \lambda\mu$ for $a \otimes \lambda \in A^F \otimes k = A$ and $x \otimes \mu \in \Psi(X)$ and noting $F_A(a \otimes \lambda) = a \otimes \lambda^q$, we obtain an $A$-module structure on $\Psi(X)$, which is clearly $F$-stable. Further, for any morphism $f : X_1 \to X_2$ in $\mathbf{mod}\text{-}A^F$, the map

$$\Psi(f) = f \otimes 1 : \Psi(X_1) \longrightarrow \Psi(X_2)$$

is obviously an $A$-module homomorphism satisfying $\Psi(f) \circ F_1 = F_2 \circ \Psi(f)$, where $F_i = F_{\Psi(X_i)}$. Hence, we obtain a functor $\Psi : \mathbf{mod}\text{-}A^F \to \mathbf{mod}^F\text{-}A$.

From the construction, we see easily from Lemma 3.1 that

$$\Psi\Phi = 1_{\mathbf{mod}^F\text{-}A} \ \text{ and } \ \Phi\Psi \cong 1_{\mathbf{mod}\text{-}A^F},$$

where $1_{\mathbf{mod}^F\text{-}A}$ and $1_{\mathbf{mod}\text{-}A^F}$ denote the identity functors of $\mathbf{mod}^F\text{-}A$ and $\mathbf{mod}\text{-}A^F$, respectively. $\square$

[1] It should be understood that the $F$'s in $A^F$ and $M^F$ are not the same.

**Corollary 3.3.** *There is a one-to-one correspondence between isoclasses of indecomposable $A^F$-modules and isoclasses of indecomposable $F$-stable $A$-modules.*

Let $(M, F_M)$ be an $F$-stable $A$-module. For each submodule $N$ of $M$ (not necessarily an $F_M$-stable subspace), the image $F_M N$ is an $A$-submodule of $M$.

**Proposition 3.4.** *Let $(M, F_M)$ and $(N, F_N)$ be two $F$-stable modules.*

(a) *Every submodule $M'$ of $M$ which is also an $F_M$-stable subspace is an $F$-stable $A$-module. In particular, both the radical $\mathrm{Rad}\, M$ and socle $\mathrm{Soc}\, M$ of $M$ are $F$-stable modules.*
(b) *As $\mathbb{F}_q$-spaces, we have $\mathrm{Hom}_A(M, N)^F \cong \mathrm{Hom}_{A^F}(M^F, N^F)$.*
(c) *We have $\mathbb{F}_q$-algebra isomorphisms $\mathrm{End}_A(M)^F \cong \mathrm{End}_{A^F}(M^F)$ and*

$$\big(\mathrm{End}_A(M)/\mathrm{Rad}\,\mathrm{End}_A(M)\big)^F \cong \mathrm{End}_{A^F}(M^F)/\mathrm{Rad}\,\mathrm{End}_{A^F}(M^F).$$

*Proof.* Since $F_M M' = M'$, the restriction of $F_M$ to $M'$ defines a Frobenius map on $M'$. So $M'$ is an $F$-stable module as the condition (3.0.1) is automatically satisfied . If $S$ is a maximal (resp. simple) submodule of $M$, so is $F_M S$. Thus, both $\mathrm{Rad}\, M$ and $\mathrm{Soc}\, M$ are submodules which are $F_M$-stable (subspaces) and hence, are $F$-stable modules. (b) is a consequence of the category isomorphism given in 3.2. The first statement in (c) follows from (b) and Lemma 2.3. We now prove the last isomorphism. We first observe that if $B$ is a semisimple algebra with Frobenius morphism $F_B$, then the fixed point algebra $B^F$ is also semisimple. Since

$$\big(\mathrm{End}_A(M)/\mathrm{Rad}\,\mathrm{End}_A(M)\big)^F \cong (\mathrm{End}_A(M))^F/(\mathrm{Rad}\,\mathrm{End}_A(M))^F,$$

it remains to prove that

$$(\mathrm{Rad}\,\mathrm{End}_A(M))^F = \mathrm{Rad}\,(\mathrm{End}_A(M))^F.$$

The inclusion "$\supseteq$" follows from the semisimplicity of the right hand side of the above isomorphism, while the inverse inclusion "$\subseteq$" follows from the fact that $\big(\mathrm{Rad}\,\mathrm{End}_A(M)\big)^F$ is nilpotent. $\square$

An $A$-module $M$ is called *$F$-periodic*, if there exists an $F$-stable $A$-module $\tilde{M}$ such that $M$ is isomorphic to a direct summand of $\tilde{M}$ (denoted $M \mid \tilde{M}$). We shall see in the next section that for a finite dimensional algebra $A$ with Frobenius morphism $F_A$ every $A$-module is $F$-periodic. However, in example 4.6, we shall see that, for an infinite dimensional algebra, there are modules which are not $F$-periodic.

We end this section with the following example which is important in sections 6–10.

**Example 3.5.** Let $Q = (Q_0, Q_1)$ be a finite quiver without loops, where $Q_0$ resp. $Q_1$ denotes the set of vertices resp. arrows of $Q$ . For each arrow $\rho$ in $Q_1$, we denote by $h\rho$ and $t\rho$ the head and the tail of $\rho$, respectively. Let $\sigma$ be an automorphism of $Q$, that is, $\sigma$ is a permutation on the vertices of $Q$ and on the arrows of $Q$ such that $\sigma(h\rho) = h\sigma(\rho)$ and $\sigma(t\rho) = t\sigma(\rho)$ for any $\rho \in Q_1$. We further assume, following [23, 12.1.1], that $\sigma$ is admissible, that is, there are no arrows connecting vertices in the same orbit of $\sigma$ in $Q_0$. We shall call the pair $(Q, \sigma)$ an *admissible quiver*, or simply an *ad-quiver*.

Let $A := kQ$ be the path algebra of $Q$ over $k = \overline{\mathbb{F}}_q$ with identity $1 = \sum_{i \in Q_0} e_i$ where $e_i$ is the idempotent (or the length 0 path) corresponding to the vertex $i$. Then $\sigma$ induces a Frobenius morphism (cf. Lemma 2.1)

$$F_{Q,\sigma} = F_{Q,\sigma;q} : A \to A;\ \sum_s x_s p_s \longmapsto \sum_s x_s^q \sigma(p_s), \tag{3.5.1}$$

where $\sum_s x_s p_s$ is a $k$-linear combination of paths $p_s$, and $\sigma(p_s) = \sigma(\rho_t) \cdots \sigma(\rho_1)$ if $p_s = \rho_t \cdots \rho_1$ for arrows $\rho_1, \cdots, \rho_t$ in $Q_1$. We shall investigate the structure of $A^F$ in §6.

## 4. Twisting modules with Frobenius maps

Let $M$ be an $A$-module and $F_M : M \to M$ a Frobenius map on the space $M$. Note that $M$ is not necessarily $F$-stable. We define its *(external) Frobenius twist* (with respect to the Frobenius morphism $F_A$ on $A$) to be the $A$-module $M^{[F_M]}$ such that $M^{[F_M]} = M$ as vector spaces with $F$-twisted action

$$a * m := F_M\big(F_A^{-1}(a)F_M^{-1}(m)\big) \text{ for all } a \in A, m \in M.$$

If $\pi : A \to \mathrm{End}_k(M)$ and $\pi^{[F_M]} : A \to \mathrm{End}_k(M^{[F_M]})$ denote the corresponding representations, then

$$\pi^{[F_M]}(a) = F_{(M,M)}(\pi(F_A^{-1}(a))) = F_M \circ \pi(F_A^{-1}(a)) \circ F_M^{-1} \text{ for all } a \in A,$$

where $F_{(M,M)}$ is the induced Frobenius map on $\mathrm{End}_k(M)$ (cf. 2.4).

**Lemma 4.1.** *Up to isomorphism, the Frobenius twist $M^{[F_M]}$ is independent of the choice of the Frobenius map $F_M$ on $M$.*

*Proof.* If $F_M$ and $F'_M$ are two Frobenius maps on $M$, then the linear isomorphism $f := F'_M \circ F_M^{-1} : M \to M$ is clearly an $A$-module isomorphism from $M^{[F_M]}$ to $M^{[F'_M]}$. □

By the lemma, we shall denote $M^{[F_M]}$ and $\pi^{[F_M]}$ by $M^{[1]}$ and $\pi^{[1]}$, respectively. Similarly, we define $M^{[-1]}$ to be the $A$-module given by $\pi^{[-1]} : A \to \mathrm{End}_k(M)$ where

$$\pi^{[-1]}(a) = F_M^{-1} \circ \pi F_A(a) \circ F_M \text{ for all } a \in A. \tag{4.1.1}$$

Inductively, for each integer $s > 1$, we define $M^{[s]} = (M^{[s-1]})^{[1]}$ and $M^{[-s]} = (M^{[-s+1]})^{[-1]}$ with respect to the *same* given $F_M$.

**Corollary 4.2.** *The Frobenius twist $(\ )^{[1]}$ defines a category isomorphism from* **mod**-$A$ *onto itself.*

*Proof.* Given two $A$-modules $M$ and $N$ with Frobenius maps $F_M$ and $F_N$, respectively, and an $A$-module homomorphism $f : M \to N$, it can be checked by using the corresponding representations that the linear map $F_{(M,N)}(f)$ defined before Lemma 2.3 is in fact an $A$-module homomorphism from $M^{[1]}$ to $N^{[1]}$. We shall denote this morphism by $f^{[1]}$. Thus, we obtain a functor $(\ )^{[1]}$ : **mod**-$A \to$ **mod**-$A$. This functor is clearly invertible with inverse $(\ )^{[-1]}$. □

If $(M, F_M)$ be an $F$-stable $A$-module and $N$ is a submodule of $M$ (not necessarily an $F_M$-stable space), then the $A$-submodule $F_M N$ of $M$ is called the "*internal*" Frobenius twist of $N$. Note that $F_M N$ is isomorphic to the (external) Frobenius twist $N^{[1]}$ of $N$ with respect to any given Frobenius map $F_N$ on $N$. This is deduced from the fact that the $k$-linear map $\varphi = F_M|_N \circ F_N^{-1} : N^{[1]} \to F_M N$ is an $A$-module isomorphism. Recall that $[M]$ denotes the isoclass of $M$.

**Proposition 4.3.** *Let $M, M_1$ and $M_2$ be $A$-modules with Frobenius maps $F_M, F_1$ and $F_2$, respectively.*

(a) *$(M, F_M)$ is $F$-stable if and only if, as $A$-modules, $M^{[1]} = M$.*
(b) *$M^{[1]} \cong M$ if and only if there exists $M' \in [M]$ such that $M' = M$ as a vector space and $(M', F_M)$ is $F$-stable.*
(c) *For any given integer $s$, $M_1 \cong M_2$ if and only if $M_1^{[s]} \cong {M_2}^{[s]}$.*

*Proof.* The statement (a) follows directly from the definition, since $(M, F_M)$ is $F$-stable if and only if $\pi \circ F_A = F_{(M,M)} \circ \pi$ which is equivalent to $\pi = \pi^{[F_M]}$ .

We now prove (b). Let $F_M$ be the Frobenius map on $M$ which defines $M^{[1]}$. Suppose there exists an $f \in GL(M)$ such that $f \circ \pi^{[1]}(a) = \pi(a) \circ f$ for all $a \in A$. By Lang-Steinberg's theorem, there exists $g \in GL(M)$ such that $f = g^{-1} \circ F(g)$, where $F$ is the restriction of $F_{(M,M)}$ on $\mathrm{End}_k(M)$ to $GL(M)$ . Since $\pi^{[1]}(a) = F_{(M,M)}(\pi(F_A^{-1}(a)))$, it follows that $F(g) \circ F_{(M,M)}(\pi(F_A^{-1}(a))) \circ F(g^{-1}) =$

$g \circ \pi(a) \circ g^{-1}$. Putting $a = F_A(b)$, we obtain $F_{(M,M)}(\pi'(b)) = \pi'(F_A(b))$, where $\pi' : A \to \mathrm{End}_k(M)$ is the representation defined by $\pi'(a) = g \circ \pi(a) \circ g^{-1}$ for all $a \in A$. Thus, the module $M'$ (with the same space as $M$) defined by $\pi'$ is the required one.

To prove (c), by induction, it suffices to prove the cases for $s = \pm 1$. We only prove the case $s = 1$; the proof for the case $s = -1$ is similar. Let $\pi_1$ and $\pi_2$ be the representations corresponding to $A$-modules $M_1$ and $M_2$, respectively. Let $\varphi : M_1 \to M_2$ be a $k$-linear isomorphism. Then $\psi = F_2 \circ \varphi \circ F_1^{-1}$ is also a $k$-linear isomorphism, where $F_i$ is a Frobenius map on $M_i$. Then, $\varphi$ is an $A$-module isomorphism if and only if $\varphi \circ \pi_1(a) = \pi_2(a) \circ \varphi$ for all $a \in A$. Clearly, the latter is equivalent to $\psi \circ \pi_1^{[1]}(a) = {\pi_2}^{[1]}(a) \circ \psi$ which means that $\psi : M_1^{[1]} \to M_2^{[1]}$ is an $A$-module isomorphism. □

We now characterize $F$-periodic modules defined at the end of last section by Frobenius twisting.

**Theorem 4.4.** *An $A$-module $M$ is $F$-periodic if and only if $M^{[r]} \cong M$ for some integer $r$.*

*Proof.* Suppose $M$ is $F$-periodic. Then so is every direct summand of $M$. Since $M^{[r]} \cong M$ implies $M^{[rs]} \cong M$ for all $s \geqslant 1$, it suffices to prove the case when $M$ is indecomposable.

If $M$ is $F$-periodic and indecomposable, then there is an $F$-stable indecomposable $A$-module $(N, F_N)$ such that $M \mid N$. Thus, $F_N^n M \mid N$ for all $n \geqslant 1$. By the Krull-Remak-Schmidt theorem, there must be a number $r$ such that $F_N^r M \cong M$, i.e., $M^{[r]} \cong M$.

Conversely, suppose $M^{[r]} \cong M$. By Proposition 4.3(b), there exists $M' \in [M]$ such that $(M', F_M)$ is an $F^r$-stable module (with respect to the Frobenius morphisms $F_A^r$ on $A$ and $F_M^r$ on $M'$), i.e., as an $A$-module ${M'}^{[r]} = M'$. Let $\pi : A \to \mathrm{End}_k(M')$ be the corresponding representation. Then $\pi^{[r]} = \pi$. Set

$$N = M' \oplus {M'}^{[1]} \oplus \cdots \oplus {M'}^{[r-1]}$$

and define a Frobenius map $F_N : N \to N$ by

$$F_N(x_0, x_1, \cdots, x_{r-1}) = (F_M(x_{r-1}), F_M(x_0), \cdots, F_M(x_{r-2})). \tag{4.4.1}$$

Since $\pi^{[r]} = \pi$, it follows that, for any $a \in A$ and $m = (x_0, x_1, \cdots, x_{r-1}) \in N$

$$\begin{aligned} F_N(am) &= F_N(\pi(a)x_0, \pi^{[1]}(a)x_1, \cdots, \pi^{[r-1]}(a)x_{r-1}) \\ &= (\pi^{[r]}(F_A(a))F_M(x_{r-1}), \pi^{[1]}(F_A(a))F_M(x_0), \cdots, \pi^{[r-1]}(F_A a)F_M(x_{r-2})) \\ &= F_A(a)F_N(m). \end{aligned}$$

Therefore, $N$ is $F$-stable and, consequently, $M$ is $F$-periodic. □

**Corollary 4.5.** *Let $A$ be finite dimensional. Then every $A$-module is $F$-periodic.*

*Proof.* Let $M$ be an $A$-module with Frobenius map $F$. By Lemma 2.1, there are $k$-basis $\{a_1, a_2, \cdots a_s\}$ of $A$ and $k$-basis $\{m_1, m_2, \cdots, m_t\}$ of $M$ such that

$$F_A(\sum_{i=1}^{s} x_i a_i) = \sum_{i=1}^{s} x_i^q a_i \ \text{ and } \ F(\sum_{j=1}^{t} y_j m_j) = \sum_{j=1}^{t} y_j^q m_j$$

for all $a = \sum_{i=1}^s x_i a_i \in A$ and all $m = \sum_{j=1}^t y_j m_j$ in $M$. Write $a_i m_j = \sum_{l=1}^t z_{ijl} m_l$ with $z_{ijl} \in k$. Since $k = \overline{\mathbb{F}}_q$, there is an integer $n$ such that all $z_{ijl}$'s lie in $\mathbb{F}_{q^n}$. Thus, $F^n(a_i m_j) = a_i m_j$ for all $i$

and $j$. Consequently, we have

$$F^n(am) = \sum_{i,j} F^n(x_i y_j a_i m_j) = \sum_{i,j} x_i^{q^n} y_j^{q^n} F^n(a_i m_j)$$
$$= (\sum_{i=1}^{s} x_i^{q^n} a_i)(\sum_{j=1}^{t} y_j^{q^n} m_j) = F_A^n(a)F^n(m).$$

This proves that $M$ is $F^n$-stable. Now the result follows from Prop. 4.3(a) and the theorem above. □

Recall that, for a path algebra $A$ of a quiver $Q$, an $A$-module can be identified as a representation $(V, \phi)$ of $Q$ where $V = \{V_i\}_{i\in Q_0}$ is a set of finite dimensional vector spaces $V_i$ and $\phi = \{\phi_\rho\}_{\rho\in Q_1}$ is a set of linear transformations $\phi_\rho : V_{t\rho} \to V_{h\rho}$.

**Example 4.6.** Let $Q$ be the quiver with two vertices 1 and 2 and with infinite many arrows from 1 to 2 indexed by $\rho_{ij}$ for all $i \geqslant 1$ and $1 \leqslant j \leqslant i$. That is, they are

$$\rho_{11}, \rho_{21}, \rho_{22}, \rho_{31}, \rho_{32}, \rho_{33}, \cdots .$$

Then the path algebra $A = kQ$ is infinite dimensional, but has the identity $1 = e_1 + e_2$. Consider the automorphism $\sigma$ of $Q$ fixing two vertices and cyclicly permuting each subset $\{\rho_{ij} | 1 \leqslant j \leqslant i\}$ of arrows for each fixed $i \geqslant 1$. Then $\sigma$ induces a Frobenius morphism $F = F_{Q,\sigma}$ on $A$ (see 3.5). Define a representation $V$ such that $V_1 = V_2 = k$ and that $\phi_{\rho_{i1}}$ is the identity on $k$, but $\phi_{\rho_{ij}} = 0$ for all $j \neq 1$. Then $V$ is two-dimensional and $V^{[r]} \ncong V$ for all $r \geqslant 0$. Therefore, $V$ is not $F$-periodic.

## 5. $F$-periods and indecomposable $F$-stable modules

Let $A$ be a $k$-algebra with a fixed Frobenius morphism $F_A$. The proof of Theorem 4.4 actually suggests a construction of indecomposable $F$-stable modules from $F$-periodic indecomposable ones.

Let $M$ be $F$-periodic with respect to a given Frobenius map $F_M$, and let $r$ be the minimal integer such that $M^{[r]} \cong M$. We shall call $r$ the *$F$-period* of $M$, denoted $p(M) = p_F(M)$. Clearly, if $M^{[s]} \cong M$, then $p(M) \mid s$ and, by Lemma 4.1, $p(M)$ is independent of the choice of $F_M$.

For $r = p(M)$, by Proposition 4.3(b), there is a new $A$-module structure $M'$ on the vector space $M$ such that $M' \cong M$ and $(M', F_M)$ is $F_M^r$-stable, that is, $M'^{[r]} = M'$. Thus, by Proposition 4.3(c), $M', M'^{[1]}, \cdots, M'^{[r-1]}$ are pairwise non-isomorphic. Let

$$\tilde{M} = M' \oplus M'^{[1]} \oplus \cdots \oplus M'^{[r-1]} \tag{5.0.1}$$

and define a Frobenius map $F_{\tilde{M}} : \tilde{M} \to \tilde{M}$ as in (4.4.1). Then $(\tilde{M}, F_{\tilde{M}})$ is $F$-stable, that is, $\tilde{M} \in \mathrm{Ob}(\mathbf{mod}^F\text{-}A)$. Thus, $\tilde{M}^F := \tilde{M}^{F_{\tilde{M}}}$ is an $A^F$-module. By Lemma 3.1 we infer that, up to isomorphism, $\tilde{M}^F$ is independent of the choice of $M'$.

The following result generalizes Kac's result [19, Lemma 3.4] for the path algebras of quivers.

**Theorem 5.1.** *Maintain the notation above. Let $M$ be an $F$-periodic indecomposable $A$-module with $F$-period $r$. Then $(\tilde{M}, F_{\tilde{M}})$ is indecomposable in $\mathbf{mod}^F$-$A$ and*

$$\mathrm{End}_{A^F}(\tilde{M}^F)/\mathrm{Rad}\,(\mathrm{End}_{A^F}(\tilde{M}^F)) \cong \mathbb{F}_{q^r}.$$

*Moreover, every indecomposable $A^F$-module is isomorphic to a module of the form $\tilde{M}^F$ for some $F$-periodic indecomposable module $M$.*

*Proof.* The Frobenius map $F_{\tilde{M}} : \tilde{M} \to \tilde{M}$ induces a Frobenius map $\tilde{F} = F_{(\tilde{M},\tilde{M})}$ on $\mathrm{End}_A(\tilde{M})$. By Lemma 3.4, we have an $\mathbb{F}_q$-algebra isomorphism

$$\mathrm{End}_{A^F}(\tilde{M}^F)/\mathrm{Rad}\,(\mathrm{End}_{A^F}(\tilde{M}^F)) \cong \left(\mathrm{End}_A(\tilde{M})/\mathrm{Rad}\,(\mathrm{End}_A(\tilde{M}))\right)^{\tilde{F}}.$$

For each $f \in \mathrm{End}_A(\tilde{M})$, we write $f$ in matrix form as

$$f = (f_{ji})_{r\times r} : \tilde{M} = \bigoplus_{i=0}^{r-1} M'^{[i]} \longrightarrow \bigoplus_{i=0}^{r-1} M'^{[i]} = \tilde{M},$$

where $f_{ji} : M'^{[i]} \to M'^{[j]}$. Then $\tilde{F}(f) = (g_{ji})$ with $g_{ji} = f^{[1]}_{j-1,i-1}$, where the indices are integers modulo $r$ . In particular, if $\tilde{F}(f) = f$, i.e., $f \in \mathrm{End}_A(\tilde{M})^{\tilde{F}}$, then $f_{ji} = f^{[1]}_{j-1,i-1}$ for all $0 \leqslant i, j \leqslant r-1$. Since $M', M'^{[1]}, \cdots, M'^{[r-1]}$ are pairwise non-isomorphic indecomposable $A$-modules, we have an algebra isomorphism

$$\begin{aligned} \mathrm{End}_A(\tilde{M})/\mathrm{Rad}\,(\mathrm{End}_A(\tilde{M})) &\longrightarrow \prod_{i=0}^{r-1} \mathrm{End}_A(M'^{[i]})/\mathrm{Rad}\,(\mathrm{End}_A(M'^{[i]})) \\ \bar{f} = (\bar{f}_{ji}) &\longmapsto (\bar{f}_{00}, \bar{f}_{11} \cdots, \bar{f}_{r-1,r-1}). \end{aligned}$$

Since $\mathrm{End}_A(M'^{[i]})/\mathrm{Rad}\,(\mathrm{End}_A(M'^{[i]})) \cong k$ for each $0 \leqslant i \leqslant r-1$, we obtain

$$\mathrm{End}_A(\tilde{M})/\mathrm{Rad}\,(\mathrm{End}_A(\tilde{M})) \cong \underbrace{k \times \cdots \times k}_{r} =: U.$$

The Frobenius map $\tilde{F}$ on the left hand side induces a Frobenius map $\tilde{F}$ on $U$ given by

$$\tilde{F}(x_0, x_1, \cdots, x_{r-1}) = (x^q_{r-1}, x^q_0, \cdots, x^q_{r-2}).$$

Hence,

$$\mathrm{End}_{A^F}(\tilde{M}^F)/\mathrm{Rad}\,(\mathrm{End}_{A^F}(\tilde{M}^F)) \cong U^{\tilde{F}} \cong \mathbb{F}_{q^r}.$$

Conversely, let $X$ be an indecomposable $A^F$-module such that

$$\mathrm{End}_{A^F}(X)/\mathrm{Rad}\,(\mathrm{End}_{A^F}(X)) \cong \mathbb{F}_{q^r}.$$

Then $X_k := X \otimes_{\mathbb{F}_q} k$ is an $F$-stable $A$-module with the Frobenius map $F := F_{X_k}$ defined by

$$F(x \otimes \lambda) = x \otimes \lambda^q \ \text{ for } x \otimes \lambda \in X \otimes_{\mathbb{F}_q} k.$$

Moreover, we have a decomposition

$$X_k = M_1 \oplus \cdots \oplus M_r,$$

where $M_1, \cdots, M_r$ are pairwise non-isomorphic indecomposable $A$-modules. The $F_A$-stability of $X_k$ implies that

$$FM_1 \oplus \cdots \oplus FM_r = FX_k = X_k = M_1 \oplus \cdots \oplus M_r.$$

Thus, the set $\Omega := \{M_1, \cdots, M_r\}$ is $F$-stable (up to isomorphism). We claim that $\Omega$ contains only one $F$-orbit. Let $\Omega_1, \cdots, \Omega_m$ be the orbits of $\Omega$. For each $1 \leqslant j \leqslant m$, let

$$\hat{M}_j = \bigoplus_{N \in \Omega_j} N.$$

Then

$$\hat{M}_j = N_j \oplus FN_j \oplus \cdots \oplus F^{r_j-1}N_j,$$

where $N_j \in \Omega_j$ and $r_j = |\Omega_j|$. Note that $F^{r_j}N_j \cong N_j$. Let $F_j$ be a Frobenius map on $N_j$. Then $FN_j$ is isomorphic to the Frobenius twist $N_j^{[1]}$ of $N_j$ with respect to $F_j$. Hence, we have $N_j^{[r_j]} \cong N_j$ and

$$\hat{M}_j \cong N_j \oplus N_j^{[1]} \oplus \cdots \oplus N_j^{[r_j-1]} \cong \hat{M}_j^{[1]}.$$

Then, by Prop. 4.3(b), we obtain an $F$-stable module $(\tilde{M}_j, \tilde{F}_j)$ satisfying $\tilde{M}_j \cong \hat{M}_j$. The Frobenius maps $\tilde{F}_j$ induces a Frobenius map $F'$ on $\oplus_{j=1}^m \tilde{M}_j$ defined by

$$F'(v_1, \cdots, v_m) = (\tilde{F}_1(v_1), \cdots, \tilde{F}_m(v_m)).$$

By Lemma 3.1, the isomorphism $X_k \cong \oplus_{j=1}^m \tilde{M}_j$ implies that

$$X = {X_k}^F \cong (\oplus_{j=1}^m \tilde{M}_j)^{F'} = \oplus_{j=1}^m \tilde{M}_j^{\tilde{F}_j}.$$

Since $X$ is indecomposable, we must have $m = 1$ and the required isomorphism

$$X_k = \hat{M}_1 \cong N_1 \oplus N_1^{[1]} \oplus \cdots \oplus N_1^{[r-1]}.$$

□

From the proof, we obtain the following correspondence.

**Corollary 5.2.** *If $A$ is finite dimensional, then there is a one-to-one correspondence between the isoclasses of indecomposable $A^F$-modules and the $F$-orbits of the isoclasses of indecomposable $A$-modules.*

A finite dimensional algebra $B$ over a field is called representation-finite if there are only finitely many isoclasses of (finite dimensional) indecomposable $B$-modules. Theorem 5.1 and Cor. 4.5 imply immediately the following.

**Corollary 5.3.** *Let $A$ be finite dimensional and $F_A$ a Frobenius morphism on $A$. Then, $A$ is representation-finite if and only if so is $A^F$.*

An $A^F$-module $X$ is called absolutely indecomposable if $X \otimes_{\mathbb{F}_q} k$ is an indecomposable $A$-module.

**Corollary 5.4.** *An $A^F$-module $X$ is absolutely indecomposable if and only if there is an $F$-stable indecomposable $A$-module $M = (M, F)$ such that $X \cong M^F$.*

## 6. Finite dimensional hereditary algebras

The first application of our theory is to show that every finite dimensional hereditary (basic) algebra over a finite field is isomorphic to the $F$-fixed point algebra of the path algebra of a finite ad-quiver (see Example 3.5). Thus, the representation theory of a finite dimensional hereditary algebra is completely determined by the counterpart of the corresponding ad-quivers.

We first recall the notion of modulated quivers (cf. [8] and [2, 4.1.9]).

**Definition 6.1.** A valued graph is a graph without loops together with a positive integer $d_x$ for each vertex $x$ and a pair of positive integers $({}_xc_y^\gamma, {}_yc_x^\gamma)$ for each edge $x \overset{\gamma}{\text{—}} y$ satisfying ${}_xc_y^\gamma d_y = {}_yc_x^\gamma d_x$. A valued graph together with an orientation is called a *valued quiver*, and a valued quiver is called *simple* if it has no parallel arrows. (Thus, opposite arrows between two vertices are allowed in a simple valued quiver.) A *modulation*[2] $\mathbb{M}$ of a valued quiver consists of an assignment of a division ring $D_x$ to each vertex $x$, and a $D_y$-$D_x$-bimodule $M_\rho$ to each arrow $\rho = x \longrightarrow y$ satisfying

(1) $\mathrm{Hom}_{D_y}(M_\rho, D_y) \cong \mathrm{Hom}_{D_x}(M_\rho, D_x)$,
(2) $\dim_{D_y}(M_\rho) = {}_xc_y^\gamma$, $\dim_{D_x}(M_\rho) = {}_yc_x^\gamma$.

Finally, a *modulated quiver* consists of a valued quiver $\Gamma$ and a modulation $\mathbb{M}$.

A modulated quiver is simple if its underlying quiver is simple.

[2]If we follow the definition given in [2, 4.1.9], then $M_\rho = \begin{cases} {}_xM_y^\gamma, & \text{if } \rho = y \xrightarrow{\gamma} x \\ {}_yM_x^\gamma, & \text{if } \rho = x \xrightarrow{\gamma} y. \end{cases}$

Let $\mathcal{Q} = (\Gamma, \mathbb{M})$ be a modulated quiver with $\Gamma_0$ (resp. $\Gamma_1$) the set of vertices (resp. arrows) of $\Gamma$ and $\mathbb{M} = (\{D_x\}_{x\in\Gamma_0}, \{M_\rho\}_{\rho\in\Gamma_1})$. Let $R = \oplus_{x\in\Gamma_0} D_x$ and $M = \oplus_{\rho\in\Gamma_1} M^\rho$. Then $M$ is a natural $R$-$R$-bimodules. The $R$-algebra

$$T(\mathcal{Q}) := \bigoplus_{n\geqslant 0} M^{\otimes n} \text{ where } M^{\otimes 0} = R, M^{\otimes n} = \underbrace{M \otimes_R \cdots \otimes_R M}_{n}$$

is called the *path* (or *tensor*) algebra of $\mathcal{Q}$. Thus, a tensor $x_n \otimes \cdots \otimes x_1$ with $x_i \in M_{\rho_i}$ is non-zero implies that $\rho_n \cdots \rho_1$ is a path in $\Gamma$.

For a modulated quiver $\mathcal{Q} = (\Gamma, \mathbb{M})$, let $\bar{\mathcal{Q}} = (\bar\Gamma, \bar{\mathbb{M}})$ be the associated simple modulated quiver obtained by summing the valuations and bimodules over parallel arrows. More precisely, $\bar\Gamma$ is a simple valued quiver defined by setting $\bar\Gamma_0 = \Gamma$, $\bar\Gamma_1 = \{\bar\rho : x \to y\}$ where $\bar\rho := \{$ all $\rho : x \to y$ in $\Gamma\}$, and setting the valuation for the arrow $\bar\rho : x \to y$ to be $(c_{\bar\rho}, c'_{\bar\rho})$ where $c_{\bar\rho} = \sum_{\rho\in\bar\rho} {}_x c^\rho_y$ and $c'_{\bar\rho} = \sum_{\rho\in\bar\rho} {}_y c^\rho_x$. The modulation $\bar{\mathbb{M}} = (\{D_x\}_{x\in\bar\Gamma_0}, \{\bar M_{\bar\rho}\}_{\bar\rho\in\bar\Gamma_1})$ is defined by setting $\bar M_{\bar\rho} = \oplus_{\rho\in\bar\rho} M_\rho$.

**Definition 6.2.** Let $\mathcal{Q} = (\Gamma, \mathbb{M})$ and $\mathcal{Q}' = (\Gamma', \mathbb{M}')$ be two modulated quiver. We say that $\mathcal{Q} \cong \mathcal{Q}'$ if there exists a quiver isomorphism $\tau : \bar\Gamma \overset{\sim}{\longrightarrow} \bar\Gamma'$ such that (1) $D_x \cong D'_{\tau(x)}$ as division rings, and (2) $M_{\bar\rho} \cong M'_{\tau(\bar\rho)}$ as bimodules via (1).

Clearly, if $\mathcal{Q} \cong \mathcal{Q}'$ then we have algebra isomorphism $T(\mathcal{Q}) \cong T(\mathcal{Q}')$.

We now construct a modulated quiver from an ad-quiver. Given a finite ad-quiver $(Q, \sigma)$, let $I = \Gamma_0$ and $\Gamma_1$ denote the set of $\sigma$-orbits in $Q_0$ and $Q_1$, respectively. Thus, we obtain a new quiver $\Gamma = (\Gamma_0, \Gamma_1)$. For each arrow $\boldsymbol{\rho} : \mathbf{i} \longrightarrow \mathbf{j}$ in $\Gamma$, define

$$\varepsilon_{\boldsymbol\rho} = \#\{\text{arrows in } \boldsymbol\rho\},\ d_{\boldsymbol\rho} = \varepsilon_{\boldsymbol\rho}/\varepsilon_{\mathbf{j}},\ \text{ and } d'_{\boldsymbol\rho} = \varepsilon_{\boldsymbol\rho}/\varepsilon_{\mathbf{i}}, \tag{6.2.1}$$

where $\varepsilon_{\mathbf{k}} = \#\{\text{vertices in } \sigma\text{-orbit } \mathbf{k}\}$ for $\mathbf{k} \in I$. The quiver $\Gamma$ together with the valuation $\{\varepsilon_{\mathbf{d}}\}_{\mathbf{d}\in\Gamma_0}$, $\{(d_{\boldsymbol\rho}, d'_{\boldsymbol\rho})\}_{\boldsymbol\rho\in\Gamma_1}$ defines a valued quiver $\Gamma = \Gamma(Q, \sigma)$. Clearly, each valued quiver can be obtained in this way from an ad-quiver.

Using the Frobenius morphism $F = F_{Q,\sigma}$ on $A$ defined in (3.5.1), we can attach naturally to $\Gamma$ an $\mathbb{F}_q$-modulation to obtain a modulated quiver (i.e., an $\mathbb{F}_q$-species) as follows: for each vertex $\mathbf{i} \in I$ and each arrow $\boldsymbol\rho$ in $\Gamma$, we fix $i_0 \in \mathbf{i}$, $\rho_0 \in \boldsymbol\rho$, and consider the $F_A$-stable subspaces of $A$

$$A_{\mathbf{i}} = \bigoplus_{i\in\mathbf{i}} ke_i = \bigoplus_{s=0}^{\varepsilon_{\mathbf{i}}-1} ke_{\sigma^s(i_0)} \text{ and } A_{\boldsymbol\rho} = \bigoplus_{\rho\in\boldsymbol\rho} k\rho = \bigoplus_{t=0}^{\varepsilon_{\boldsymbol\rho}-1} k\sigma^t(\rho_0),$$

where $e_i$ denotes the idempotent corresponding to the vertex $i$. Then

$$A^F_{\mathbf{i}} = \{\sum_{s=0}^{\varepsilon_{\mathbf{i}}-1} x^{q^s} e_{\sigma^s(i_0)} \mid x \in k, x^{q^{\varepsilon_{\mathbf{i}}}} = x\} \text{ and } A^F_{\boldsymbol\rho} = \{\sum_{t=0}^{\varepsilon_{\boldsymbol\rho}-1} x^{q^t} \sigma^t(\rho_0) \mid x \in k, x^{q^{\varepsilon_{\boldsymbol\rho}}} = x\}. \tag{6.2.2}$$

Further, the algebra structure of $A$ induces an $A^F_{\mathbf{j}}$-$A^F_{\mathbf{i}}$-bimodule structure on $A^F_{\boldsymbol\rho}$ where $\boldsymbol\rho : \mathbf{i} \longrightarrow \mathbf{j}$. Thus, we obtain an $\mathbb{F}_q$-modulation $\mathbb{M} = \mathbb{M}(Q, \sigma) := (\{A^F_{\mathbf{i}}\}_{\mathbf{i}}, \{A^F_{\boldsymbol\rho}\}_{\boldsymbol\rho})$ over the valued quiver $\Gamma$. We shall denote the $\mathbb{F}_q$-modulated quiver $\mathcal{Q} = (\Gamma, \mathbb{M})$ defined above by

$$\mathfrak{M}_{Q,\sigma} = \mathfrak{M}_{Q,\sigma;q} = (\Gamma, \mathbb{M}). \tag{6.2.3}$$

Let $T(\mathfrak{M}_{Q,\sigma})$ be the tensor algebra of the modulated quiver $\mathfrak{M}_{Q,\sigma}$. Thus, by definition, $T(\mathfrak{M}_{Q,\sigma}) = \oplus_{n\geqslant 0} M^{\otimes n}$, where $M = \oplus_{\boldsymbol\rho\in\Gamma_1} A^F_{\boldsymbol\rho}$ is viewed as an $R$-$R$-bimodule with $R = \oplus_{\mathbf{i}\in I} A^F_{\mathbf{i}}$ and $\otimes = \otimes_R$. If, for each $\sigma$-orbit $\mathbf{p}$ of a path $\rho_n \cdots \rho_2\rho_1$ in $Q$, we set $A_{\mathbf{p}} = \oplus_{p\in\mathbf{p}} kp$. Then

$$A^F_{\mathbf{p}} \cong A^F_{\boldsymbol\rho_n} \otimes_{\mathbb{F}_{n-1}} \cdots \otimes_{\mathbb{F}_2} A^F_{\boldsymbol\rho_2} \otimes_{\mathbb{F}_1} A^F_{\boldsymbol\rho_1},$$

where $\boldsymbol{\rho}_t$ is the $\sigma$-orbit of $\rho_t$ and $\mathbb{F}_t = A^F_{h\boldsymbol{\rho}_t}$. Since $A^F = \oplus_{\mathbf{p}} A^F_{\mathbf{p}}$, it follows that the fixed point algebra $A^F$ is isomorphic to the tensor algebra $T(\mathfrak{M}_{Q,\sigma})$. Thus, $A^F$-modules can be identified with representations of the modulated quiver $\mathfrak{M}_{Q,\sigma}$ (see [8]). The above observation together with Theorem 3.2 implies the following.

**Proposition 6.3.** *Let $(Q,\sigma)$ be an ad-quiver with path algebra $A = kQ$ and induced Frobenius morphism $F = F_{Q,\sigma}$. Let $\mathfrak{M}_{Q,\sigma}$ be the associated $\mathbb{F}_q$-modulated quiver defined as above.*

(a) *We have an algebra isomorphism $A^F \cong T(\mathfrak{M}_{Q,\sigma})$. Hence the categories $\mathbf{mod}^F$-$A$ and $\mathbf{mod}$-$T(\mathfrak{M}_{Q,\sigma})$ are equivalent.*
(b) *If $Q$ has no oriented cycles, then the fixed-point algebra $A^F$ is a finite dimensional hereditary basic algebra.*

**Corollary 6.4.** *Maintain the notation above and let $r \geqslant 1$ be an integer. Then, the ad-quiver $(Q,\sigma^r)$ defines an $\mathbb{F}_{q^r}$-modulated quiver $\mathfrak{M}_{Q,\sigma^r;q^r}$ whose tensor algebra is isomorphic to the $\mathbb{F}_{q^r}$-algebra $A^F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$.*

*Proof.* Clearly, $F_{Q,\sigma^r;q^r} = F^r_{Q,\sigma;q} = F^r$. By Lemma 2.1, we have $A = A^F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and $F(a \otimes \lambda) = a \otimes \lambda^q$. Then $A^{F^r} = A^F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$. Now, the isomorphism follows from Prop. 6.3(a). □

Note that for $r > 1$ the modulated quiver $\mathfrak{M}_{Q,\sigma^r;q^r}$ is different from the modulated quivers $\mathfrak{M}_{Q,\sigma^r;q}$ and $\mathfrak{M}_{Q,\sigma;q^r}$. The former has the same underlying valued quiver as $\mathfrak{M}_{Q,\sigma^r;q^r}$, but different base field, while the latter has the same base field but different underlying valued quiver (if $\sigma \neq 1$).

Our next result shows that the converse of Prop. 6.3 (b) is also true.

**Theorem 6.5.** *Let $B$ be a finite dimensional hereditary basic algebra over $\mathbb{F}_q$. Then there is an ad-quiver $(Q,\sigma)$ such that $B$ is isomorphic to $(kQ)^{F_{Q,\sigma}}$.*

*Proof.* Let $A = B \otimes k$ and define $F : A \to A$ by $F(b \otimes \lambda) = b \otimes \lambda^q$. Clearly, $F$ is a Frobenius morphism on the $k$-algebra $A$ and $A^F = B$. Since $B$ is a finite dimensional hereditary basic algebra, it follows that so is $A$ and $B$ is isomorphic to the tensor (or path) algebra of the associated modulated Ext-quiver $Q_B$ (see [2, p.104]). In particular, we have algebra isomorphism

$$\varphi : B \xrightarrow{\sim} \operatorname{gr} B \quad \text{where} \quad \operatorname{gr} B = \oplus_{i \geqslant 0} \operatorname{Rad}^i B / \operatorname{Rad}^{i+1} B.$$

Since $\operatorname{Rad} A = \operatorname{Rad}(B \otimes k) = (\operatorname{Rad} B) \otimes k$ (see, e.g., [3, p.146]), we have by induction $\operatorname{Rad}^i A = (\operatorname{Rad}^i B) \otimes k$ for all $i \geqslant 1$. Thus, the isomorphism $\varphi$ induces a $k$-algebra isomorphism

$$\tilde{\varphi} : A = B \otimes k \xrightarrow{\sim} \operatorname{gr} A = \operatorname{gr} B \otimes k.$$

Let $\bar{F}$ denote the Frobenius morphism on $\operatorname{gr} A$ induced from $F$. Clearly, $\bar{F}$ stabilizes each direct summand $\operatorname{Rad}^i A / \operatorname{Rad}^{i+1} A$ of $\operatorname{gr} A$ and $B = A^F \cong (\operatorname{gr} A)^{\bar{F}}$.

We now prove that $(\operatorname{gr} A, \bar{F})$ defines an ad-quiver $(Q,\sigma)$ such that $(\operatorname{gr} A)^{\bar{F}} \cong (kQ)^{F_{Q,\sigma}}$. Suppose

$$A/\operatorname{Rad} A = k\bar{e}_1 \oplus \cdots \oplus k\bar{e}_n$$

where $e_i$ are primitive orthogonal idempotents of $A$ with $1 = e_1 + \cdots + e_n$. Since the $F(e_i)$'s form a complete set of primitive orthogonal idempotents of $A$, it follows that there is a permutation $\sigma$ and an invertible element $u \in A$ such that $F(e_i) = u e_{\sigma i} u^{-1}$. Thus, $\bar{F}(\bar{e}_i) = \bar{e}_{\sigma i}$ for all $i$ .

Let $Q_0 = \{1, 2, \cdots, n\}$ and $I$ the set of $\sigma$-orbits. Putting $f_{\mathbf{i}} = \sum_{i \in \mathbf{i}} e_i$ for each $\mathbf{i} \in I$, we have

$$\operatorname{Rad} A/\operatorname{Rad}^2 A = \bigoplus_{\mathbf{i},\mathbf{j} \in I} f_{\mathbf{i}} (\operatorname{Rad} A/\operatorname{Rad}^2 A) f_{\mathbf{j}},$$

where

$$f_{\mathbf{i}} (\operatorname{Rad} A/\operatorname{Rad}^2 A) f_{\mathbf{j}} = \bar{f}_{\mathbf{i}} (\operatorname{Rad} A/\operatorname{Rad}^2 A) \bar{f}_{\mathbf{j}} = \bigoplus_{i \in \mathbf{i}, j \in \mathbf{j}} \bar{e}_i (\operatorname{Rad} A/\operatorname{Rad}^2 A) \bar{e}_j.$$

Clearly, $\bar F$ stabilizes each summand $f_{\mathbf{i}}(\operatorname{Rad} A/\operatorname{Rad}^2 A)f_{\mathbf{j}}$ and $\bar F(V_{ij}) = V_{\sigma i,\sigma j}$ where

$$V_{ij} = \bar e_i(\operatorname{Rad} A/\operatorname{Rad}^2 A)\bar e_j.$$

Fix $i, j$ and let $s$ be the smallest integer such that $\sigma^s i = i$ and $\sigma^s j = j$. Since $\bar F^s$ stabilizes $V_{ij}$, we can choose a $k$-basis $v_1, \cdots, v_{t_{ij}}$ of $V_{ij}$ such that $\bar F^s(v_a) = v_a$ for all $1 \leqslant a \leqslant t_{ij}$. Thus, we obtain a $k$-basis

$$\{v_1, \cdots, v_t, \bar F(v_1), \cdots, \bar F(v_t), \cdots, \bar F^{s-1}(v_1), \cdots, \bar F^{s-1}(v_t)\}$$

for the $\bar F$-stable space $\oplus_{a=0}^{s-1} V_{\sigma^a i,\sigma^a j}$. Clearly, such a basis can be constructed for every $\sigma$-orbit of the set $\{(i, j) \mid i \in \mathbf{i}, j \in \mathbf{j}\}$. Thus, we obtain a basis $\{v_a^{ij}\}_{i,j,a}$ for $f_{\mathbf{i}}(\operatorname{Rad} A/\operatorname{Rad}^2 A)f_{\mathbf{j}}$, and hence for $\operatorname{Rad} A/\operatorname{Rad}^2 A$, which is $\bar F$-stable, i.e., $\{\bar F(v_a^{ij})\}_{i,j,a} = \{v_a^{ij}\}_{i,j,a}$. If $Q_1$ is the set of arrows $\rho_a^{ij}$ indexed by the basis elements $v_a^{ij}$, then $\bar F$ induces a permutation $\sigma$ on $Q_1$.

So we have obtained an ad-quiver $(Q, \sigma)$. The standard $k$-algebra isomorphism $\psi : kQ \to \operatorname{gr} A$ sending $i$ to $\bar e_i$ and $\rho_a^{ij}$ to $v_a^{ij}$ is compatible with the Frobenius morphisms $\bar F$ and $F_{Q,\sigma}$, that is, $\psi \circ F_{Q,\sigma} = \bar F \circ \psi$. Consequently, we obtain

$$B \cong (\operatorname{gr} A)^{\bar F} \cong (kQ)^{F_{Q,\sigma}}.$$

$\square$

**Corollary 6.6.** *Every finite dimensional hereditary algebra over a finite field is Morita equivalent to the $F$-fixed point algebra of the path algebra of an ad-quiver.*

**Remarks 6.7.** (a) If we identify $A_{\mathbf{i}}^F$ and $A_{\boldsymbol\rho}^F$ defined in (6.2.2) with $\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}$ and $\mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}}$ via

$$\sum_{i\in\mathbf{i}} x_i e_i \longmapsto x_{i_0} \quad\text{and}\quad \sum_{\rho\in\boldsymbol\rho} y_\rho \rho \longmapsto y_{\rho_0},$$

respectively, where $i_0 \in \mathbf{i}$ and $j_0 \in \mathbf{j}$ are fixed, then, for an arrow $\boldsymbol\rho : \mathbf{i} \to \mathbf{j}$, the induced $\mathbb{F}_{q^{\varepsilon_{\mathbf{j}}}}$-$\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}$-bimodule structure on $A_{\boldsymbol\rho} = \mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}}$ is not necessarily the natural bimodule ${}_{\mathbb{F}_{q^{\varepsilon_{\mathbf{j}}}}}(\mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}})_{\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}}$ induced by the subfield structure. This is because, for $r, s \geqslant 1$, the $\mathbb{F}_q$-algebra isomorphism

$$\mathbb{F}_{q^r} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^s} \cong \underbrace{\mathbb{F}_{q^m} \times \cdots \times \mathbb{F}_{q^m}}_{d},$$

means that there are exactly $d$ non-isomorphic simple $\mathbb{F}_{q^r}$-$\mathbb{F}_{q^s}$-bimodules. Here $d$ and $m$ denote the greatest common divisor and the least common multiple of $r, s$, respectively. More precisely, for $0 \leqslant r \leqslant \varepsilon_{\mathbf{i}} - 1$ and $0 \leqslant s \leqslant \varepsilon_{\mathbf{j}} - 1$ with $\rho_0 : \sigma^r(i_0) \to \sigma^s(j_0)$, the induced $\mathbb{F}_{q^{\varepsilon_{\mathbf{j}}}}$-$\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}$-bimodule structure on $A_{\boldsymbol\rho} = \mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}}$ is given by

$$y \cdot z \cdot x = y^{q^s} z x^{q^r} \quad\text{for}\quad x \in \mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}, y \in \mathbb{F}_{q^{\varepsilon_{\mathbf{j}}}}, z \in \mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}},$$

while the natural bimodule structure corresponds only to the case when $r = s = 0$. These two bimodule structures on $\mathbb{F}_{q^{\varepsilon_{\boldsymbol\rho}}}$ are not necessarily isomorphic.

(b) The $\mathbb{F}_q$-modulated quiver studied in [15], [17], and [18] involve only the natural bimodules ${}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^n})_{\mathbb{F}_{q^s}}$, where $r, s, n \geqslant 1$, $r \mid n$ and $s \mid n$. These will be called *natural modulated quivers* below. Note that the natural $\mathbb{F}_{q^r}$-$\mathbb{F}_{q^s}$-bimodule ${}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^n})_{\mathbb{F}_{q^s}}$ is isomorphic to the direct sum of $\frac{n}{m}$ copies of the natural simple bimodule ${}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^m})_{\mathbb{F}_{q^s}}$. Though non-isomorphic bimodules involved in two modulated quivers could result in isomorphic tensor algebras, the following example shows that not every finite dimensional hereditary basic algebra over $\mathbb{F}_q$ arises from a natural modulated quiver.

**Example 6.8.** Let $\Gamma$ denote the valued quiver

$$\underset{a}{\bullet} \xrightarrow{\ \rho\ } \underset{b}{\bullet} \xrightarrow{\ \tau\ } \underset{c}{\bullet}$$

with $(\varepsilon_a, \varepsilon_b, \varepsilon_c) = (1, 2, 2)$, $(d_\rho, d'_\rho) = (1, 2)$, and $(d_\tau, d'_\tau) = (2, 2)$. Then the pair of natural bimodules ${}_{\mathbb{F}_q}(\mathbb{F}_{q^2})_{\mathbb{F}_{q^2}}$ and ${}_{\mathbb{F}_{q^2}}(\mathbb{F}_{q^4})_{\mathbb{F}_{q^2}}$ together with the valued quiver $\Gamma$ defines the natural $\mathbb{F}_q$-modulated quiver $\mathfrak{M}$. Let $M$ denote the natural bimodule ${}_{\mathbb{F}_q^2}(\mathbb{F}_{q^2})_{\mathbb{F}_{q^2}}$ and $M' = \mathbb{F}_{q^2}$ (as $\mathbb{F}_q$-vector spaces) denote the $\mathbb{F}_{q^2}$-$\mathbb{F}_{q^2}$-bimodule given by $x \cdot y \cdot z = xyz^q$. It is easy to see that $M$ and $M'$ are not isomorphic as $\mathbb{F}_{q^2}$-$\mathbb{F}_{q^2}$-bimodules. The pair of bimodules ${}_{\mathbb{F}_q}(\mathbb{F}_{q^2})_{\mathbb{F}_{q^2}}$ and $M \oplus M'$ also defines a new $\mathbb{F}_q$-modulated quiver $\mathfrak{M}'$ whose tensor algebra $T(\mathfrak{M}')$ is not isomorphic to the tensor algebra $T(\mathfrak{M})$ of $\mathfrak{M}$, since $T(\mathfrak{M}) \otimes_{\mathbb{F}_q} k$ and $T(\mathfrak{M}') \otimes_{\mathbb{F}_q} k$ are respectively isomorphic to the path algebras of the following quivers

and are obviously not isomorphic.

## 7. Almost split sequences

The Auslander-Reiten theory is one of the fundamental tools in the study of representations of algebras (see, e.g., [1]). This and next sections are devoted to establishing a relation between the Auslander-Reiten theories of $A$ and its fixed-point algebra $A^F$.

We briefly review the general theory. Let $A$ be a finite dimensional algebra over an *arbitrary* field $k$. A morphism $\varphi : L \to M$ in **mod**-$A$ is called *minimal right almost split* if

(a) $\varphi$ is not a split epimorphism,
(b) any morphism $X \to M$ which is not a split epimorphism factors through $\varphi$,
(c) any morphism $f : L \to L$ satisfying $\varphi = \varphi f$ is an isomorphism.

It is easy to see that, if $\varphi : L \to M$ is a minimal right almost split morphism, then $M$ is indecomposable.

A *minimal left almost split* morphism is defined dually.

A short exact sequence $0 \to N \xrightarrow{\psi} L \xrightarrow{\varphi} M \to 0$ is called an *almost split sequence* if $\varphi$ is minimal right almost split, or equivalently, if $\psi$ is minimal left almost split.

**Theorem 7.1.** *(Auslander-Reiten) Let $M$ be an indecomposable $A$-module.*

(a) *There exists a unique, up to isomorphism, minimal right almost split morphism $\varphi : L \to M$. If, moreover, $M$ is not projective, then the sequence*
$$0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow L \xrightarrow{\varphi} M \longrightarrow 0$$
*is an almost split sequence.*

(b) *There exists a unique, up to isomorphism, minimal left almost split morphism $\psi : M \to L$. If, moreover, $M$ is not injective, then the sequence*
$$0 \longrightarrow M \xrightarrow{\psi} L \longrightarrow \operatorname{Coker}\psi \longrightarrow 0$$
*is an almost split sequence.*

Given an almost split sequence $0 \to N \xrightarrow{\psi} L \xrightarrow{\varphi} M \to 0$, the module $N$ (resp. $M$) is uniquely (up to isomorphism) determined by $M$ (resp. $N$). We write $\tau_A M = N$ or $N = \tau_A^{-1} M$. The $\tau_A =: \tau$ is called the *Auslander-Reiten translation* of $A$, which indeed admits the following description using the transpose and the dual (see [1, Chap. IV, V]. Let $\mathcal{P}$ and $\mathcal{I}$ be the ideals of **mod**-$A$ (in the sense of [13, p.16]) defined respectively by
$$\mathcal{P}(X, Y) = \{f : X \to Y | f \text{ factors through a projective } A\text{-module}\}$$

and

$$\mathcal{I}(X,Y) = \{f : X \to Y | f \text{ factors through an injective } A\text{-module}\},$$

where $X, Y \in \textbf{mod-}A$. The factor categories of $\textbf{mod-}A$ by $\mathcal{P}$ and $\mathcal{I}$ are denoted by $\underline{\textbf{mod}}\text{-}A$ and $\overline{\textbf{mod}}\text{-}A$, respectively. Let $M \in \textbf{mod-}A$ and let

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective presentation of $M$. Applying $\mathrm{Hom}_A(-, A)$, we get a morphism

$$f^* := \mathrm{Hom}_A(f, A) : \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A)$$

in $\textbf{mod-}A^{\mathrm{op}}$. Thus, we obtain an $A^{\mathrm{op}}$-module $\mathrm{Coker}\, f^*$ which is called the *transpose* of $M$ and denoted by $\mathrm{Tr}\, M$. In general, Tr does not give rise to a functor from $\textbf{mod-}A$ to $\textbf{mod-}A^{\mathrm{op}}$. However, Tr induces a functor $\mathrm{Tr} : \underline{\textbf{mod}}\text{-}A \to \underline{\textbf{mod}}\text{-}A^{\mathrm{op}}$ such that the compositions $\underline{\textbf{mod}}\text{-}A \overset{\mathrm{Tr}}{\to} \underline{\textbf{mod}}\text{-}A^{\mathrm{op}} \overset{\mathrm{Tr}}{\to} \underline{\textbf{mod}}\text{-}A$ and $\underline{\textbf{mod}}\text{-}A^{\mathrm{op}} \overset{\mathrm{Tr}}{\to} \underline{\textbf{mod}}\text{-}A \overset{\mathrm{Tr}}{\to} \underline{\textbf{mod}}\text{-}A^{\mathrm{op}}$ are isomorphic to the identity on $\underline{\textbf{mod}}\text{-}A$ and $\underline{\textbf{mod}}\text{-}A^{\mathrm{op}}$, respectively. On the other hand, the duality $D = \mathrm{Hom}_k(-, k) : \textbf{mod-}A \to \textbf{mod-}A^{\mathrm{op}}$ induces a duality $D : \underline{\textbf{mod}}\text{-}A \to \overline{\textbf{mod}}\text{-}A^{\mathrm{op}}$ such that the composition $D\,\mathrm{Tr} : \underline{\textbf{mod}}\text{-}A \to \overline{\textbf{mod}}\text{-}A$ is an equivalence with inverse equivalence $\mathrm{Tr}\, D : \overline{\textbf{mod}}\text{-}A \to \underline{\textbf{mod}}\text{-}A$. Then, for each indecomposable non-projective $A$-module $M$, we have $\tau M \cong D\,\mathrm{Tr}\, M$, and for each indecomposable non-injective $A$-module $N$, we have $\tau^{-1} N \cong \mathrm{Tr}\, DN$.

For any two $A$-modules $M = \oplus_i M_j$ and $N = \oplus_j N_j$, where $M_i$ and $N_j$ are indecomposable, we define the *radical* of $\mathrm{Hom}_A(M, N)$ by

$$\mathrm{Rad}_A(M, N) = \{(f_{ji}) : M \to N \mid f_{ji} : M_i \to N_j \text{ is not an isomorphism for all } i, j\}.$$

In fact, $\mathrm{Rad}_A(-, -)$ is an ideal of $\textbf{mod-}A$. Inductively, for each $n > 1$, the $n$-th power of the radical is defined to be

$$\mathrm{Rad}_A^n(M, N) = \sum_X \mathrm{Rad}_A^{n-1}(X, N) \circ \mathrm{Rad}_A(M, X).$$

From the functorial point of view (see, e.g., [2, 4.8]), we may study almost split sequences in the category $\textbf{Fun}(A)$ (resp. $\textbf{Fun}^{\mathrm{op}}(A)$) whose objects are the covariant (resp. contravariant) additive functors from $\textbf{mod-}A$ (resp. $\textbf{mod-}A^{\mathrm{op}}$) to the category $\textbf{vec}_k$ of $k$-vector spaces, and whose morphisms are the natural transformations of functors. Thus, each $A$-module $M$ defines a functor $\mathrm{Hom}_A(-, M)$ in $\textbf{Fun}^{\mathrm{op}}(A)$ by $\mathrm{Hom}_A(-, M)(N) = \mathrm{Hom}_A(N, M)$ and a subfunctor $\mathrm{Rad}_A(-, M)$ by $\mathrm{Rad}_A(-, M)(N) = \mathrm{Rad}_A(N, M)$. We denote the induced quotient functor by

$$\mathcal{H}_M := \mathrm{Hom}_A(-, M)/\mathrm{Rad}_A(-, M).$$

In particular, if $M$ is indecomposable, $\mathcal{H}_M$ is a simple functor — a functor that has no non-zero proper subfunctor. Dually, each $A$-module $M$ defines a functor $\mathrm{Hom}_A(M, -)$ in $\textbf{Fun}(A)$ and the quotient functor $\mathcal{K}_M$ by its subfunctor $\mathrm{Rad}_A(M, -)$. We have the following functorial characterization of minimal right or left almost split morphisms; see [12, 1.4] or [2, 4.12.6].

**Proposition 7.2.** (a) *A morphism $\varphi : L \to M$ of $A$-modules is minimal right almost split if and only if the induced sequence*

$$0 \longrightarrow \mathrm{Hom}_A(-, \ker\varphi) \xrightarrow{\kappa_*} \mathrm{Hom}_A(-, L) \xrightarrow{\varphi_*} \mathrm{Hom}_A(-, M) \xrightarrow{\xi_M} \mathcal{H}_M \longrightarrow 0$$

*is a minimal projective resolution of $\mathcal{H}_M$ in $\textbf{Fun}^{\mathrm{op}}(A)$, where $\kappa : \mathrm{Ker}\,\varphi \to L$ is the canonical inclusion and $\xi_M$ denotes the canonical projection.*

(b) *A morphism $\psi : N \to L$ of $A$-modules is minimal left almost split if and only if the induced sequence*

$$0 \longrightarrow \mathrm{Hom}_A(\mathrm{Coker}\,\psi, -) \xrightarrow{\pi_*} \mathrm{Hom}_A(L, -) \xrightarrow{\psi_*} \mathrm{Hom}_A(N, -) \xrightarrow{\xi_N} \mathcal{K}_N \longrightarrow 0$$

*is a minimal projective resolution of* $\mathcal{K}_N$ *in* $\mathbf{Fun}(A)$*, where* $\pi : L \to \operatorname{Coker}\psi$ *is the canonical projection.*

*We now assume* $k = \overline{\mathbb{F}}_q$ and let $F_A$ be a fixed Frobenius morphism on $A$. We also assume that each $A$-module $M$ has an $\mathbb{F}_q$-structure given by a Frobenius map $F_M$.

**Lemma 7.3.** *A morphism* $\varphi : L \to M$ *(resp.* $\psi : N \to L$*) in* $\mathbf{mod}$-$A$ *is minimal right (resp. left) almost split if and only if so is* $\varphi^{[1]} : L^{[1]} \to M^{[1]}$ *(resp.* $\psi^{[1]} : N^{[1]} \to L^{[1]}$*). In other words, a sequence* $0 \to N \xrightarrow{\psi} L \xrightarrow{\varphi} M \to 0$ *is an almost split sequence if and only if so is* $0 \to N^{[1]} \xrightarrow{\psi^{[1]}} L^{[1]} \xrightarrow{\varphi^{[1]}} M^{[1]} \to 0$*. Moreover,* $\tau M^{[1]} \cong (\tau M)^{[1]}$*.*

*Proof.* Since the Frobenius twisting functor $(\ )^{[1]}$ is invertible with the inverse $(\ )^{[-1]}$ given in (4.1.1), everything is clear. $\square$

Let $\varphi : L \to M$ be a minimal right almost split morphism in $\mathbf{mod}$-$A$. Then, by the lemma above, for each integer $s \geqslant 1$, $\varphi^{[s]} : L^{[s]} \to M^{[s]}$ is minimal right almost split. Let $r = p(M)$ be the $F$-period of $M$, i.e., $r$ is minimal with $M^{[r]} \cong M$. Then Theorem 7.1 implies $L^{[r]} \cong L$. By Proposition 4.3(b), we may assume that $M^{[r]} = M$ and $L^{[r]} = L$. Thus, both $\tilde{M} = \oplus_{i=0}^{r-1} M^{[i]}$ and $\tilde{L} = \oplus_{i=0}^{r-1} L^{[i]}$ defined in (5.0.1) are $F$-stable with respect to Frobenius maps $F_{\tilde{M}}$ and $F_{\tilde{L}}$ defined by $F_M$ and $F_L$, respectively (see (4.4.1)). Since both $\varphi : L \to M$ and $\varphi^{[r]} : L \to M$ are minimal right almost split, there is a $g \in \operatorname{Aut}_A(L)$ such that $\varphi = \varphi^{[r]} \circ g$. Let $F = F_{(L,L)}$ and $F_{(L,M)}$ be the induced Frobenius maps on $\operatorname{Hom}_A(L, L)$ and $\operatorname{Hom}_A(L, M)$, respectively; see Lemma 2.3. Then $F_{(L,M)}(\psi \circ f) = F_{(L,M)}(\psi) \circ F(f)$ for all $\psi \in \operatorname{Hom}_A(L, M)$ and $f \in \operatorname{End}_A(L)$. Restricting $F$ to the connected algebraic group $\operatorname{Aut}_A(L)$ and applying Lang-Steinberg's theorem, we may find an element $h \in \operatorname{Aut}_A(L)$ satisfying $g = F^r(h)h^{-1}$, that is, $h = g^{-1}F^r(h)$. This implies that

$$\varphi \circ h = \varphi \circ (g^{-1}F^r(h)) = \varphi^{[r]} \circ F^r(h) = F^r_{(L,M)}(\varphi \circ h).$$

Note that, by Cor. 4.2, $F^r_{(L,M)}(\varphi) = \varphi^{[r]}$. Thus, $\varphi \circ h : L \to M$ is minimal right almost split satisfying $(\varphi \circ h)^{[r]} = \varphi \circ h$. Replacing $\varphi$ by $\varphi \circ h$, we may assume that $\varphi$ is chosen to satisfy $\varphi^{[r]} = \varphi$. Now we define

$$\tilde{\varphi} := \operatorname{diag}(\varphi, \varphi^{[1]}, \cdots, \varphi^{[r-1]}) : \tilde{L} = \bigoplus_{i=0}^{r-1} L^{[i]} \to \bigoplus_{i=0}^{r-1} M^{[i]} = \tilde{M}.$$

The equality $\varphi^{[r]} = \varphi$ implies that $\tilde{\varphi}$ is a morphism in $\mathbf{mod}^F$-$A$. Hence, $\tilde{\varphi}$ induces an $A^F$-module morphism $\tilde{\varphi}^F : \tilde{L}^F \to \tilde{M}^F$. Here again we drop the subscripts of the $F$'s for notational simplicity.

**Theorem 7.4.** *Let* $\varphi : L \to M$ *be a minimal right almost split morphism. Then there exists an induced morphism* $\tilde{\varphi} : \tilde{L} \to \tilde{M}$ *of* $F$*-stable modules such that its restriction* $\tilde{\varphi}^F : \tilde{L}^F \to \tilde{M}^F$ *is a minimal right almost split morphism in* $\mathbf{mod}$-$A^F$*. In particular, every almost split sequence* $0 \to N \xrightarrow{\psi} L \xrightarrow{\varphi} M \to 0$ *in* $\mathbf{mod}$-$A$ *gives rise to an almost split sequence*

$$0 \longrightarrow \tilde{N}^F \xrightarrow{\tilde{\psi}^F} \tilde{L}^F \xrightarrow{\tilde{\varphi}^F} \tilde{M}^F \longrightarrow 0$$

*in* $\mathbf{mod}$-$A^F$*. Moreover, every almost split sequence of* $A^F$*-modules can be constructed in this way.*

*Proof.* Clearly, the last assertion follows from Theorem 5.1. From the construction of $\tilde{\varphi}$ above, we have $\operatorname{Ker}\tilde{\varphi} = \oplus_{i=0}^{r-1} \operatorname{Ker}\varphi^{[i]}$ and the Frobenius map $F_{\tilde{L}}$ on $\tilde{L}$ induces a Frobenius map on $\operatorname{Ker}\tilde{\varphi}$. Thus, $\operatorname{Ker}\tilde{\varphi}$ is $F$-stable.

For each $0 \leqslant i \leqslant r-1$, by Prop. 7.2 and Lemma 7.3, the morphism $\varphi^{[i]} : L^{[i]} \to M^{[i]}$ gives the minimal projective resolution of $\mathcal{H}_{M^{[i]}}$

$$0 \longrightarrow \mathrm{Hom}_A(-, \mathrm{Ker}\,\varphi^{[i]}) \xrightarrow{\kappa_*^{[i]}} \mathrm{Hom}_A(-, L^{[i]}) \xrightarrow{\varphi_*^{[i]}} \mathrm{Hom}_A(-, M^{[i]}) \xrightarrow{\xi_{M^{[i]}}} \mathcal{H}_{M^{[i]}} \longrightarrow 0,$$

where $\kappa$ is the inclusion $\mathrm{Ker}\,\tilde{\varphi}^{[i]} \to L^{[i]}$. Summing up, we obtain a minimal projective resolution of $\mathcal{H}_{\tilde{M}} \cong \oplus_{i=0}^{r-1} \mathcal{H}_{M^{[i]}}$

$$0 \longrightarrow \mathrm{Hom}_A(-, \mathrm{Ker}\,\tilde{\varphi}) \xrightarrow{\tilde{\kappa}_*} \mathrm{Hom}_A(-, \tilde{L}) \xrightarrow{\tilde{\varphi}_*} \mathrm{Hom}_A(-, \tilde{M}) \xrightarrow{\xi_{\tilde{M}}} \mathcal{H}_{\tilde{M}} \longrightarrow 0,$$

where $\tilde{\kappa} = \mathrm{diag}\{\kappa_i\} : \mathrm{Ker}\,\tilde{\varphi} = \oplus_{i=0}^{r-1} \mathrm{Ker}\,\varphi^{[i]} \to \tilde{L}$. Thus, for each $F$-stable module $(X, F_X)$, we get the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(X, \mathrm{Ker}\,\tilde{\varphi}) \xrightarrow{\tilde{\kappa}_*(X)} \mathrm{Hom}_A(X, \tilde{L}) \xrightarrow{\tilde{\varphi}_*(X)} \mathrm{Hom}_A(X, \tilde{M}) \xrightarrow{\xi_{\tilde{M}}(X)} \mathcal{H}_{\tilde{M}}(X) \longrightarrow 0.$$

Now the Frobenius maps on modules induce Frobenius maps $\tilde{F}$ on each space in the sequence above. It is easy to see that all morphisms in the sequence are compatible with those Frobenius maps $\tilde{F}$. This gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(X, \mathrm{Ker}\,\tilde{\varphi})^{\tilde{F}} \longrightarrow \mathrm{Hom}_A(X, \tilde{L})^{\tilde{F}} \longrightarrow \mathrm{Hom}_A(X, \tilde{M})^{\tilde{F}} \longrightarrow \mathcal{H}_{\tilde{M}}(X)^{\tilde{F}} \longrightarrow 0.$$

Then we deduce from Prop. 3.4 the following exact sequence

$$0 \to \mathrm{Hom}_{A^F}(X^F, \mathrm{Ker}\,(\tilde{\varphi}^F)) \to \mathrm{Hom}_{A^F}(X^F, \tilde{L}^F) \to \mathrm{Hom}_{A^F}(X^F, \tilde{M}^F) \to \mathcal{H}_{\tilde{M}^F}(X^F) \to 0.$$

Since every $A^F$-module is of the form $X^F$ (Thm 3.2), we obtain a minimal projective resolution of the simple functor $\mathcal{H}_{\tilde{M}^F}$ in $\mathbf{Fun}(A^F)$

$$0 \longrightarrow \mathrm{Hom}_{A^F}(-, \mathrm{Ker}\,(\tilde{\varphi}^F)) \xrightarrow{\tilde{\kappa}_*^F} \mathrm{Hom}_{A^F}(-, \tilde{L}^F) \xrightarrow{\tilde{\varphi}_*^F} \mathrm{Hom}_{A^F}(-, \tilde{M}^F) \xrightarrow{\xi_{\tilde{M}}^F} \mathcal{H}_{\tilde{M}^F} \longrightarrow 0.$$

Therefore, by Lemma 7.2 again, $\tilde{\varphi}^F : \tilde{L}^F \to \tilde{M}^F$ is a minimal right almost split morphism in $\mathbf{mod}$-$A^F$. □

Dually, for each minimal left almost split morphism $\psi : N \to L$ in $\mathbf{mod}$-$A$, we can construct a minimal left almost split morphism $\tilde{\psi}^F : \tilde{N}^F \to \tilde{L}^F$ in $\mathbf{mod}$-$A^F$ in a similar way. We leave the detail to the reader.

## 8. The Auslander-Reiten quivers

We are now going to prove that the Frobenius morphism $F$ on $A$ induces an automorphism $\mathfrak{s}$ of the Auslander-Reiten quiver $\mathcal{Q}$ of $A$ and that the induced modulated quiver $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ is essentially the Auslander-Reiten quiver of $A^F$.

We begin with the general definition. Let $A$ be a finite dimensional algebra over an *arbitrary* field $k$. For an $A$-module $M$, let $D_M$ denote the $k$-algebra

$$D_M := \mathrm{End}_A(M)/\mathrm{Rad}\,(\mathrm{End}_A(M)).$$

This is a division algebra if $M$ is indecomposable. By definition, the *Auslander-Reiten quiver* (or *AR-quiver* for short) of $A$ is a (simple) $k$-modulated quiver $\mathcal{Q}_A$ consisting of a valued graph $\Gamma = \Gamma_A$ and a $k$-modulation $\mathbb{M} = \mathbb{M}_A$ defined on $\Gamma$. Here, the vertices of $\Gamma$ are isoclasses $[M]$ of indecomposable $A$-modules and the arrows $[M] \to [N]$ for indecomposable $M$ and $N$ are defined by the condition $\mathrm{Irr}_A(M, N) \neq 0$, where

$$\mathrm{Irr}_A(M, N) := \mathrm{Rad}\,_A(M, N)/\mathrm{Rad}\,_A^2(M, N)$$

is the space of irreducible homomorphisms from $M$ to $N$. Each arrow $[M] \to [N]$ has the valuation $(d_{MN}, d'_{MN})$ with $d_{MN}$ and $d'_{MN}$ being the dimensions of $\mathrm{Irr}_A(M, N)$ considered as left $D_N$-space

and right $D_M$-space, respectively. The $k$-modulation $\mathbb{M}$ is given by division algebras $D_M$ for vertices $[M]$ and (non-zero) $D_N$-$D_M$-bimodules $\mathrm{Irr}_A(M,N)$ for arrows $[M] \to [N]$.

**Remarks 8.1.** (a) The AR-quiver of $A$ defined in [1, VII.1] is simply the valued quiver $\Gamma_A$ together with the translation $\tau$ sending non-projective vertices to non-injective vertices. The modulation $\mathbb{M}_A$ for $\mathcal{Q}_A$ is not explicitly mentioned there. Here we adopt the definition for AR-quivers given by Benson in [2, p.150]. We will see that this definition is more natural to fit our situation. Moreover, we shall also see that the translation $\tau$ for $A^F$ is naturally induced from the translation for $A$.

(b) The valuation $(d, d')$ of an arrow $[M] \to [N]$ in $\mathcal{Q}_A$ admits the following description (see [1, VII.1]). If $L \to N$ is minimal right almost split, then $L \cong d'M \oplus L_1$, where $L_1$ has no summand isomorphic to $M$. If $M \to K$ is minimal left almost split, then $K \cong dN \oplus K_1$, where $K_1$ admits no summand isomorphic to $N$.

(c) If $k$ is algebraically closed, then $D_M \cong k$ and $d_{MN} = d'_{MN} = \dim_k \mathrm{Irr}(M,N)$ for all indecomposable $A$-modules $M$ and $N$. So the modulation in the AR-quiver $\mathcal{Q}_A$ consists of $k$-spaces which can be represented by drawing $d_{MN}$ arrows from $[M]$ to $[N]$. In this way, we turn the modulated quiver $\mathcal{Q}_A$ to an ordinary quiver.

We return to our usual setup: let $k = \overline{\mathbb{F}}_q$ and let $F_A$ be a fixed Frobenius morphism on $A$. For each indecomposable $A$-module $M$, let $r = p(M)$ be the period of $M$. In view of Prop. 4.3(b), we may assume that $M^{[r]} = M$. Then $\tilde{M} = M \oplus M^{[1]} \oplus \cdots \oplus M^{[r-1]}$ is $F$-stable with $F_{\tilde{M}} : \tilde{M} \to \tilde{M}$ defined by

$$F_{\tilde{M}}(x_0, x_1, \cdots, x_{r-1}) = (F_M(x_{r-1}), F_M(x_0), \cdots, F_M(x_{r-2})).$$

By Theorem 5.1, $\tilde{M}^F$ is an indecomposable $A^F$-module. By Prop. 3.4, the induced Frobenius map $F = F_{(\tilde{M},\tilde{M})}$ on $\mathrm{End}_A(\tilde{M})$ gives a canonical $\mathbb{F}_q$-algebra isomorphism

$$\big(\mathrm{End}_A(\tilde{M})/\mathrm{Rad}\,\mathrm{End}_A(\tilde{M})\big)^F \cong \mathrm{End}_{A^F}(\tilde{M}^F)/\mathrm{Rad}\,\mathrm{End}_{A^F}(\tilde{M}^F).$$

In particular, if $\tilde{M} = X_k := X \otimes k$ for an indecomposable $A^F$-module $X$, then

$$\big(\mathrm{End}_A(\tilde{M})/\mathrm{Rad}\,\mathrm{End}_A(\tilde{M})\big)^F = (D_{X_k})^F \cong D_X.$$

Let $X$ and $Y$ be indecomposable $A^F$-modules. Up to isomorphism, we may assume that $X = \tilde{M}^F$ and $Y = \tilde{N}^F$ for some indecomposable $A$-modules $M$ and $N$ (see the proof of Thm 5.1). In fact, we may choose $\tilde{M} = X_k$ and $\tilde{N} = Y_k$ so that $M$ and $N$ are indecomposable direct summands of the $A$-modules $X_k$ and $Y_k$, respectively. We then have

$$D_X \cong (D_{\tilde{M}})^F \cong \mathbb{F}_{q^{r_1}} \ \text{ and } \ D_Y \cong (D_{\tilde{N}})^F \cong \mathbb{F}_{q^{r_2}},$$

where $r_1 = p(M)$ and $r_2 = p(N)$. Moreover, the Frobenius map $F_{(\tilde{M},\tilde{N})}$ on $\mathrm{Hom}_A(\tilde{M},\tilde{N})$ induces Frobenius maps on $\mathrm{Rad}^n_A(\tilde{M},\tilde{N})$ for each $n \geqslant 1$, and thus, a Frobenius map $F$ on $\mathrm{Irr}_A(\tilde{M},\tilde{N})$. By viewing $\big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F$ as a $D_Y$-$D_X$-bimodules via the isomorphisms $(D_{\tilde{M}})^F \cong D_X$ and $(D_{\tilde{N}})^F \cong D_Y$, we have the following lemma.

**Lemma 8.2.** *Let $X = \tilde{M}^F$ and $Y = \tilde{N}^F$ be indecomposable $A^F$-modules, where $M$ and $N$ are indecomposable $A$-modules. Then the $D_Y$-$D_X$-bimodules $\mathrm{Irr}_{A^F}(X,Y)$ and $\big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F$ are isomorphic.*

*Proof.* It is clear to see that restriction of the linear isomorphism given in Prop. 3.4(b) gives an $\mathbb{F}_q$-linear isomorphism

$$\Psi : \mathrm{Rad}_{A^F}(X,Y) \xrightarrow{\sim} \big(\mathrm{Rad}_A(\tilde{M},\tilde{N})\big)^F; f \longmapsto f \otimes 1,$$

and hence, an $\mathbb{F}_q$-linear injection

$$\Psi : \mathrm{Rad}\,^2_{A^F}(X,Y) \to \big(\mathrm{Rad}\,^2_A(\tilde{M},\tilde{N})\big)^F.$$

Thus, $\Psi$ induces a surjective map

$$\bar{\Psi} : \mathrm{Irr}_{A^F}(X,Y) \to \big(\mathrm{Rad}\,_A(\tilde{M},\tilde{N})\big)^F / \big(\mathrm{Rad}\,^2_A(\tilde{M},\tilde{N})\big)^F \cong \big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F,$$

which is clearly a $D_Y$-$D_X$-bimodule homomorphism. We now prove that $\bar{\Psi}$ is a linear isomorphism by a comparison of dimensions.

For each $0 \leqslant s \leqslant p(M)-1$ and each $0 \leqslant t \leqslant p(N)-1$, we set $n_{st} = \dim_k \mathrm{Irr}_A(M^{[s]}, N^{[t]})$. Since

$$\mathrm{Irr}_A(\tilde{M},\tilde{N}) \cong \bigoplus_{\substack{0\leqslant s\leqslant p(M)-1\\ 0\leqslant t\leqslant p(N)-1}} \mathrm{Irr}_A(M^{[s]}, N^{[t]}),$$

we have $n := \dim_k \mathrm{Irr}_A(\tilde{M},\tilde{N}) = \sum_{s,t} n_{st}$, and so $\dim_{\mathbb{F}_q}\big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F = n$. Now, take a minimal right almost split map $L \to N$. Then, so is $L^{[t]} \to N^{[t]}$ by Lemma 7.3. For each fixed $s$, it holds that $L^{[t]} \cong n_{st}M^{[s]} \oplus L_{s,t}$ with $M^{[s]} \nmid L_{s,t}$ for all $0 \leqslant t \leqslant p(N)-1$. Thus, $\tilde{L} \cong n_s M^{[s]} \oplus L_s$, for all $s$ with $0 \leqslant s \leqslant p(M)-1$, where $n_s = \sum_{t=0}^{p(N)-1} n_{st}$ and $L_s = \oplus_{t=0}^{p(N)-1} L_{s,t}$. Since $\tilde{L} = \tilde{L}^{[1]} \cong n_s M^{[s+1]} \oplus L_s^{[1]}$ and the $M^{[s]}$ are pairwise non-isomorphic, it follows that $n_s = n_{s+1}$ for all $0 \leqslant s < p(M)-1$. Thus, $n_s = \frac{n}{p(M)}$ and $\tilde{L} = \frac{n}{p(M)}\tilde{M} \oplus \tilde{L}'$ for some $F$-stable module $\tilde{L}'$ with $\tilde{M} \nmid \tilde{L}'$.

On the other hand, by Theorem 7.4 (see also Remark 8.1(b)), $\tilde{L}^F \to \tilde{N}^F = Y$ is minimal right almost split and

$$\tilde{L}^F = \frac{n}{p(M)}\tilde{M}^F \oplus \tilde{L}'^F = \frac{n}{p(M)}X \oplus \tilde{L}'^F$$

with $X \nmid \tilde{L}'^F$, it follows that, if $(d,d')$ be the valuation of the arrow $[X] \to [Y]$ in the AR-quiver $\mathcal{Q}_{A^F}$ of $A^F$, then $d' = \frac{n}{p(M)}$. Hence,

$$\dim_{\mathbb{F}_q} \mathrm{Irr}_{A^F}(X,Y) = d' \dim_{\mathbb{F}_q} D_X = n = \dim_{\mathbb{F}_q}\big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F.$$

Consequently, $\bar{\Psi}$ is a $D_Y$-$D_X$-bimodule isomorphism. □

Since the algebra $A$ is defined over the algebraically close field $k = \overline{\mathbb{F}}_q$, we may regard the AR-quiver $\mathcal{Q} = \mathcal{Q}_A$ of $A$ as an ordinary quiver (see Remark 8.1(c)). We first observe that $\mathcal{Q}$ admits an admissible automorphism $\mathfrak{s}$. For each vertex $[M] \in \mathcal{Q}$, $\mathfrak{s}([M])$ is defined to be $[M^{[1]}]$. If $M$ and $N$ are indecomposable $A$-modules, then there are $n_{st}$ arrows $\gamma_{s,t}^{(m)}$ from $[M^{[s]}]$ to $[N^{[t]}]$ in $\mathcal{Q}$, where $0 \leqslant s \leqslant p(M)-1$, $0 \leqslant t \leqslant p(N)-1$, $n_{st} = \dim_k \mathrm{Irr}_A(M^{[s]}, N^{[t]})$ and $1 \leqslant m \leqslant n_{st}$. Note that $n_{st} = n_{s+1,t+1}$ for all $s,t$, where subscripts are considered as integers modulo $p(M)$ and $p(N)$, respectively. We now define

$$\mathfrak{s}(\gamma_{s,t}^{(m)}) = \gamma_{s+1,t+1}^{(m)} \ \text{ for all } 0 \leqslant s \leqslant p(M)-1 \text{ and } 0 \leqslant t \leqslant p(N)-1.$$

Clearly, $\mathfrak{s}$ is an admissible quiver automorphism and $(\mathcal{Q},\mathfrak{s})$ is an ad-quiver.

Associated to $(\mathcal{Q},\mathfrak{s})$, we may define a modulated quiver $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ as in (6.2.3): let $\mathcal{A} = k\mathcal{Q}$ denote the path algebra of $\mathcal{Q}$ and $F = F_{\mathcal{Q},\mathfrak{s}}$ be the Frobenius morphism of $\mathcal{A}$ induced by the automorphism $\mathfrak{s}$. For each vertex $\mathbf{i}(M)$ (i.e., the $\mathfrak{s}$-orbit of $[M]$) and each arrow $\boldsymbol{\rho}$ (i.e., an $\mathfrak{s}$-orbit of arrows in $\mathcal{Q}$) in $\Gamma(\mathcal{Q},\mathfrak{s})$, we define subspaces

$$\mathcal{A}_{\mathbf{i}(M)} = \bigoplus_{s=0}^{p(M)-1} ke_{[M^{[s]}]} \ \text{ and } \ \mathcal{A}_{\boldsymbol{\rho}} = \bigoplus_{\rho\in\boldsymbol{\rho}} k\rho,$$

of $\mathcal{A}$, which are obviously $F$-stable. By definition, the $\mathbb{F}_q$-modulation $\mathbb{M}(\mathcal{Q},\mathfrak{s})$ is given by $(\mathcal{A}_{\mathbf{i}(M)})^F$ and $(\mathcal{A}_{\boldsymbol{\rho}})^F$ for all vertices $\mathbf{i}(M)$ and arrows $\boldsymbol{\rho}$ in $\Gamma(\mathcal{Q},\mathfrak{s})$.

Recall the definition 6.2 of isomorphisms for modulated quivers. We now can state the following result.

**Theorem 8.3.** *The modulated quiver $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ associated to the AR-quiver $(\mathcal{Q},\mathfrak{s})$ of $A$ defined above is isomorphic to the AR-quiver $\mathcal{Q}_{A^F}$ of $A^F$. Moreover, the Auslander-Reiten translation of $A$ naturally induces that of the fixed-point algebra $A^F$.*

*Proof.* If $X=\tilde{M}^F$ is an indecomposable $A^F$-module, then clearly, the correspondence $[X]\longrightarrow \mathbf{i}(M)$ gives a bijection between vertices of $\mathcal{Q}_{A^F}$ and those of $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ (see Thm5.1). Moreover, there is an isomorphism

(8.3.1)
$$\mathcal{A}_{\mathbf{i}(M)}=\bigoplus_{s=0}^{p(M)-1} ke_{[M^{[s]}]}\longrightarrow \bigoplus_{s=0}^{p(M)-1} D_{M^{[s]}}\cong D_{\tilde{M}},\ \sum_s x_s e_{[M^{[s]}]}\longmapsto (x_s 1_{M^{[s]}})_s$$

which induces an $\mathbb{F}_q$-algebra isomorphism $(\mathcal{A}_{\mathbf{i}(M)})^F\cong (D_{\tilde{M}})^F\cong D_X$. As before, we shall identify $(\mathcal{A}_{\mathbf{i}(M)})^F$ with $D_X$. Thus, $(\mathcal{A}_{\boldsymbol{\rho}})^F$ is a $D_Y$-$D_X$-bimodule.

It remains to prove that for $X=\tilde{M}^F$ and $Y=\tilde{N}^F$, where $M$ and $N$ are indecomposable $A$-modules, there is a $D_Y$-$D_X$-bimodule isomorphism

$$\bigoplus_{\boldsymbol{\rho}:\mathbf{i}(M)\to\mathbf{i}(N)} (\mathcal{A}_{\boldsymbol{\rho}})^F\cong \mathrm{Irr}_{A^F}(X,Y). \tag{8.3.2}$$

By Lemma 8.2, it suffices to show that we have a $(D_{\tilde{N}})^F$-$(D_{\tilde{M}})^F$-bimodule isomorphism

$$\bigoplus_{\boldsymbol{\rho}:\mathbf{i}(M)\to\mathbf{i}(N)} (\mathcal{A}_{\boldsymbol{\rho}})^F\cong \big(\mathrm{Irr}_A(\tilde{M},\tilde{N})\big)^F,$$

or a $D_{\tilde{N}}$-$D_{\tilde{M}}$-bimodule isomorphism

$$\phi:\bigoplus_{\boldsymbol{\rho}:\mathbf{i}(M)\to\mathbf{i}(N)} \mathcal{A}_{\boldsymbol{\rho}}\xrightarrow{\sim}\mathrm{Irr}_A(\tilde{M},\tilde{N}) \tag{8.3.3}$$

which is compatible with the Frobenius morphism $F_{\mathcal{A}}$ on $\mathcal{A}$ and $F$ on $\mathrm{Irr}_A(\tilde{M},\tilde{N})$.

Let $d$ and $l$ denote the greatest common divisor and the least common multiple of $p_1=p(M)$ and $p_2=p(N)$, respectively. As before, let $n_{st}=\dim_k \mathrm{Irr}_A(M^{[s]},N^{[t]})$ and denote by $\gamma_{s,t}^{(m)}$ all arrows from $[M^{[s]}]$ to $[N^{[t]}]$ in $\Gamma(\mathcal{Q},\mathfrak{s})$. Then the set of arrows

$$\{\gamma_{s,t}^{(m)}|0\leqslant s\leqslant p_1-1, 0\leqslant t\leqslant p_2-1, 1\leqslant m\leqslant n_{st}\}$$

is $\mathfrak{s}$-stable and the arrows $\gamma_{0,t}^{(m)}$ with $0\leqslant t\leqslant d-1$ and $1\leqslant m\leqslant n_{0t}$ form a complete set of representatives of $\mathfrak{s}$-orbits in this set, each of which is of length $l$. On the other hand, for each $0\leqslant t\leqslant d-1$, the space $\mathrm{Irr}_A(M,N^{[t]})$ is $F^l$-stable with $l$ minimal. Thus, we can choose a $k$-basis $\xi_{0,t}^{(m)}$, $1\leqslant m\leqslant n_{0t}$, of $\mathrm{Irr}_A(M,N^{[t]})$ which are $F^l$-fixed. Then the set

$$\{F^a(\xi_{0,t}^{(m)})\mid 0\leqslant t\leqslant d-1, 1\leqslant m\leqslant n_{0t}, 0\leqslant a\leqslant l-1\}$$

is a $k$-basis for $\bigoplus_{s,t}\mathrm{Irr}_A(M^{[s]},M^{[t]})\cong \mathrm{Irr}_A(\tilde{M},\tilde{N})$. Thus, the correspondence $\gamma_{0,t}^{(m)}\longmapsto \xi_{0,t}^{(m)}$ induces an isomorphism of $k$-spaces

$$\phi:\bigoplus_{\boldsymbol{\rho}:\mathbf{i}(M)\to\mathbf{i}(N)} \mathcal{A}_{\boldsymbol{\rho}}=\bigoplus_{s,t,m} k\gamma_{s,t}^{(m)}\xrightarrow{\sim}\mathrm{Irr}_A(\tilde{M},\tilde{N})$$

sending $\gamma_{s,t+s}^{(m)}$ to $F^s(\xi_{0,t}^{(m)})$. Therefore, $\phi\circ F_{\mathcal{A}}=F\circ\phi$. Using the isomorphism given in (8.3.1), we see easily that $\phi$ is a $D_{\tilde{N}}$-$D_{\tilde{M}}$-bimodule isomorphism. So (8.3.3) is proved.

To prove the last assertion, we first note that the Frobenius morphism $F_A$ on $A$ is also a Frobenius morphism on $A^{\rm op}$. Since the projective cover of an $A$-module is unique up to isomorphism, each $F$-stable $A$-module $M$ admits a minimal projective presentation $P_1 \xrightarrow{f} P_0 \xrightarrow{\pi} M \longrightarrow 0$ such that $P_0$ and $P_1$ are $F$-stable projective and that $f$ and $\pi$ are morphisms in $\mathbf{mod}^F\text{-}A$. Thus the transpose ${\rm Tr} : \underline{\mathbf{mod}}\text{-}A \to \underline{\mathbf{mod}}\text{-}A^{\rm op}$ induces a functor $\mathcal{T} : \underline{\mathbf{mod}}^F\text{-}A \to \underline{\mathbf{mod}}^F\text{-}A^{\rm op}$, where, for example, $\underline{\mathbf{mod}}^F\text{-}A$ denotes the factor category of $\mathbf{mod}^F\text{-}A$ modulo morphisms factoring through an $F$-stable projective $A$-module. It is obvious that the dual $D = {\rm Hom}_k(-,k) : \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}A^{\rm op}$ induces a dual $\mathcal{D} : \underline{\mathbf{mod}}^F\text{-}A \to \overline{\mathbf{mod}}^F\text{-}A^{\rm op}$. On the other hand, for the fixed algebras $A^F$ and $(A^{\rm op})^F = (A^F)^{\rm op}$, we also have the transpose ${\rm Tr}^F : \underline{\mathbf{mod}}\text{-}A^F \to \underline{\mathbf{mod}}\text{-}(A^F)^{\rm op}$ and the dual $D^F = {\rm Hom}_{\mathbb{F}_q}(-,\mathbb{F}_q) : \underline{\mathbf{mod}}\text{-}A^F \to \overline{\mathbf{mod}}\text{-}(A^F)^{\rm op}$. Furthermore, the functor $\Phi = \Phi_A : \mathbf{mod}^F\text{-}A \to \mathbf{mod}\text{-}A^F$ defined in the proof of Theorem 3.2 induces functors $\Phi_A : \underline{\mathbf{mod}}^F\text{-}A \to \underline{\mathbf{mod}}\text{-}A^F$ and $\Phi_A : \overline{\mathbf{mod}}^F\text{-}A \to \overline{\mathbf{mod}}\text{-}A^F$. By Proposition 3.4, we finally have the following commutative square

$$\begin{tikzcd}
\underline{\mathbf{mod}}^F\text{-}A \arrow[r, "\mathcal{D}\mathcal{T}"] \arrow[d, "\Phi_A"'] & \overline{\mathbf{mod}}^F\text{-}A^{\rm op} \arrow[d, "\Phi_{A^{\rm op}}"] \\
\underline{\mathbf{mod}}\text{-}A^F \arrow[r, "D^F\,{\rm Tr}^F"] & \overline{\mathbf{mod}}\text{-}(A^{\rm op})^F
\end{tikzcd}$$

$\square$

**Remarks 8.4.** (1) By the definition of $\mathfrak{s}$, each $(\mathcal{A}_{\boldsymbol{\rho}})^F$ is a simple $D_Y$-$D_X$-bimodule. Thus, the isomorphism in (8.3.2) gives a decomposition of the $D_Y$-$D_X$-bimodule ${\rm Irr}_{A^F}(X,Y)$ into the sum of simples.

(2) By Thms 7.4 and 8.3, we see that the Auslander-Reiten theory for algebras defined over a finite field $\mathbb{F}_q$ is completely determined by the theory for algebras defined over the algebraic closure $\overline{\mathbb{F}}_q$.

## 9. Counting the number of $F$-stable representations

In this section, we shall use the theory established in §6 to count the number of representations (resp. indecomposable representations) of a finite dimensional hereditary (basic) $\mathbb{F}_q$-algebra in terms of representations of an ad-quiver. In particular, we shall prove that, for a given dimension vector, these numbers are polynomials in $q$.

Let $(Q,\sigma)$ be an ad-quiver and $\mathfrak{M}_{Q,\sigma}$ the associated modulated quiver with underlying valued quiver $\Gamma = \Gamma(Q,\sigma) = (I,\Gamma_1)$. By definition, a representation $V = (V,\phi)$ of $Q$ over $k$ consists of a $Q_0$-graded $k$-vector space $V = \oplus_{i\in Q_0} V_i$ and a family of $k$-linear maps $\phi = (\phi_\rho)_\rho$ with $\phi_\rho : V_i \to V_j$ for all arrows $\rho : i \to j \in Q_1$. A morphism from $(V,\phi)$ to $(V',\phi')$ is given by a $Q_0$-graded morphism $f = (f_i)_i : V \to V'$ such that $f_{t\rho} \circ \phi_\rho = \phi'_\rho \circ f_{h\rho}$ for each arrow $\rho$. We denote by ${\rm Rep}\, Q = {\rm Rep}\,_k Q$ the category of all finite dimensional representations of $Q$ over $k$. It is well known that $\mathbf{mod}\text{-}A$, where $A = kQ$, and ${\rm Rep}\, Q$ are isomorphic categories. For each $Q_0$-graded vector space $V = \oplus_{i\in Q_0} V_i$, we set

$$\mathbf{dim}\, V = \sum_{i\in I} (\dim_k V_i) i \in \mathbb{N}Q_0,$$

called the *dimension vector* of $V$.

Following Lusztig [24], by $\mathcal{C}_I$ we denote the sub-category of the category $\mathcal{V}_{k,\mathbb{F}_q}$ (see the definition before 2.3) whose objects are $Q_0$-graded $k$-vector spaces $V = \oplus_{i\in Q_0} V_i$ with an $\mathbb{F}_q$-rational structure determined by a Frobenius map $F = F_V : V \to V$ such that $F(V_i) = V_{\sigma(i)}$ for each $i \in Q_0$. The

morphisms in $\mathcal{C}_I$ are $k$-linear maps compatible with both the gradings and the Frobenius maps. Note that, for each $\alpha = \sum_{i\in Q_0} a_i i \in \mathbb{N}Q_0$ satisfying $a_i = a_{\sigma(i)}$, $\forall i \in Q_0$, there is a unique, up to isomorphism, object in $\mathcal{C}_I$ with dimension vector $\alpha$ (see remarks before 2.3).

A representation $(V, \phi)$ of $Q$ with $V \in \mathcal{C}_I$ is called *$F$-stable* if, for each $\alpha \in Q_1$, the equality $F \circ \phi_\rho = \phi_{\sigma(\rho)} \circ F$ holds, that is, the following diagram commutes

$$\begin{tikzcd} V_{t\rho} \arrow[r, "\phi_\rho"] \arrow[d, "F"'] & V_{h\rho} \arrow[d, "F"] \\ V_{\sigma(t\rho)} \arrow[r, "\phi_{\sigma(\rho)}"] & V_{\sigma(h\rho)} \end{tikzcd}$$

Let $\mathrm{Rep}^F(Q,\sigma)$ be the category consisting of $F$-stable representations of $Q$ together with morphisms in $\mathrm{Rep}\,Q$ which are compatible with Frobenius maps. Clearly, $\mathrm{Rep}^F(Q,\sigma)$ is a subcategory of $\mathrm{Rep}\,Q$, though not necessarily full. In fact, $\mathrm{Rep}^F(Q,\sigma)$ is an abelian $\mathbb{F}_q$-category. It is easy to see that the category isomorphism between **mod**-$A$ and $\mathrm{Rep}\,Q$ induces a category isomorphism between $\mathbf{mod}^F$-$A$, where $F = F_{Q,\sigma}$, and $\mathrm{Rep}^F(Q,\sigma)$. Hence, by Theorem 3.2, categories $\mathrm{Rep}^F(Q,\sigma)$ and **mod**-$A^F$ are isomorphic.

The quiver automorphism $\sigma$ extends linearly to a group automorphism $\sigma$ on $\mathbb{Z}Q_0$ defined by $\sigma(\sum_{i\in Q_0} a_i i) = \sum_{i\in Q_0} a_i\sigma(i)$. Clearly, if $V$ is $F$-stable, then $\sigma(\mathbf{dim}\,V) = \mathbf{dim}\,V$. Let $(\mathbb{Z}Q_0)^\sigma$ denote the $\sigma$-fixed-point subset of $\mathbb{Z}Q_0$. This set can be identified with the group $\mathbb{Z}I$ via the canonical bijection

$$\hat{\sigma} : (\mathbb{Z}Q_0)^\sigma \to \mathbb{Z}I;\ \sum_{i\in Q_0} b_i i \mapsto \sum_{\mathbf{i}\in I} a_{\mathbf{i}}\mathbf{i}, \tag{9.0.1}$$

where $a_{\mathbf{i}} := b_i = b_j$ for all $i, j \in \mathbf{i}$ (see [17, §2]). In particular, the *dimension vector* $\mathbf{dim}\,X = \sum_{\mathbf{i}\in I} d_{\mathbf{i}}\mathbf{i} \in \mathbb{N}I$ of an $A^F$-module $X$ can be defined by

$$\mathbf{dim}\,X = \hat{\sigma}(\mathbf{dim}\,(X \otimes k)).$$

Note that if, for each $\mathbf{i} \in I$, $S_{\mathbf{i}}$ denotes the simple $A^F$-module corresponding to the idempotent $e_{\mathbf{i}} = \sum_{i\in\mathbf{i}} e_i$ of $A^F$, then $d_{\mathbf{i}} = \dim_{\mathbb{F}_q^{\varepsilon_{\mathbf{i}}}} e_{\mathbf{i}}X$ is the number of composition factors isomorphic to $S_{\mathbf{i}}$ in a composition series of $X$.

Given a matrix $x = (x_{ij}) \in k^{m\times n}$ and an integer $r \geqslant 0$, we define

$$x^{[r]} = (x_{ij}^{q^r}) \in k^{m\times n}.$$

For each $\beta = \sum_{i\in Q_0} b_i i \in (\mathbb{Z}Q_0)^\sigma$, let $V_i = k^{b_i}$ for each $i \in Q_0$. We define a Frobenius map $F$ on the $Q_0$-graded vector space $V = \oplus_{i\in Q_0} V_i$ such that, for $v \in V_i$, $F(v) = v^{[1]} \in V_{\sigma(i)}$ for all $i \in Q_0$. We further define

$$R(\beta) = R(Q,\beta) = \prod_{\rho\in Q_1} \mathrm{Hom}_k(k^{b_{t\rho}}, k^{b_{h\rho}}) \cong \prod_{\rho\in Q_1} k^{b_{h\rho}\times b_{t\rho}}.$$

Then the Frobenius map $F$ on $V$ induces a Frobenius map on the variety $R(\beta)$ such that, for $x = (x_\rho) \in R(\beta)$, $F(x) = (y_\rho)$ is defined by

$$y_\rho(F(v)) = F(x_{\sigma^{-1}(\rho)}(v)) \quad \text{for all} \quad \rho \in Q_1, v \in V_{\sigma^{-1}(t\rho)}.$$

By viewing each $x_\rho$ as a matrix over $k$, we have $y_{\sigma(\rho)} = x_\rho^{[1]}$. Obviously, a point $x = (x_\rho)_\rho$ of $R(\beta)$ determines a representation $V(x) = (V_i, x_\rho)$ of $Q$. The algebraic subgroup

$$G(\beta) = \prod_{i\in Q_0} GL_{b_i}(k)$$

of $GL(V)$ acts on $R(\beta)$ by conjugation

$$(g_i)_i \cdot (x_\rho)_\rho = (g_{h\rho} x_\rho g_{t\rho}^{-1})_\rho,$$

and the $G(\beta)$-orbits $\mathcal{O}_x$ in $R(\beta)$ correspond bijectively to the isoclasses $[V(x)]$ of representations of $Q$ with dimension vector $\beta$.

The Frobenius map $F$ on $V$ also induces a Frobenius map on the group $GL(V)$ given by

$$F(gv) = F(g)F(v) \quad \text{for all} \quad g \in GL(V), v \in V.$$

It is clear that the subgroup $G(\beta)$ is $F$-stable such that $F(g)_{\sigma(i)} = g_i^{[1]}$ for $g = (g_i) \in G(\beta)$. The action of $G(\beta)$ on $R(\beta)$ restricts to an action of $G(\beta)^F$ on $R(\beta)^F$. Then, the $G(\beta)^F$-orbits in $R(\beta)^F$ correspond bijectively to the isoclasses of $F$-stable representations in $\operatorname{Rep}^F(Q,\sigma)$ with dimension vector $\hat{\sigma}(\beta)$, or equivalently, to the isoclasses of $A^F$-modules with dimension vector $\hat{\sigma}(\beta)$.

For each $\alpha = \sum_{\mathbf{i}\in I} a_{\mathbf{i}}\mathbf{i} \in \mathbb{N}I$, let $M_{Q,\sigma}(\alpha, q)$ (resp. $I_{Q,\sigma}(\alpha, q)$) be the number of isoclasses of $A^F$-modules (resp. indecomposable $A^F$-modules) of dimension vector $\alpha$. Further, as indicated above, $M_{Q,\sigma}(\alpha, q)$ is the number of $G(\beta)^F$-orbits in $R(\beta)^F$, where $\beta = \hat{\sigma}^{-1}(\alpha) \in \mathbb{N}Q_0$.

Let $\beta = \sum_{i\in Q_0} b_i i$. Note that $b_i = a_{\mathbf{i}}$ for all $i \in \mathbf{i}$. For each $\mathbf{i} \in I$ and each arrow $\boldsymbol{\rho} : \mathbf{i} \to \mathbf{j}$ in $\Gamma$, we define

$$G_{\mathbf{i}} = \prod_{i\in\mathbf{i}} GL_{b_i}(k) \;\text{ and }\; R_{\boldsymbol{\rho}} = \prod_{\rho\in\boldsymbol{\rho}} \operatorname{Hom}_k(k^{b_{t\rho}}, k^{b_{h\rho}}) \cong \prod_{\rho\in\boldsymbol{\rho}} k^{b_{h\rho}\times b_{t\rho}}.$$

By fixing an $i_0 \in \mathbf{i}$ and a $\rho_0 \in \boldsymbol{\rho}$, we can identify

$$G_{\mathbf{i}}^F = \{(g_i)_{i\in\mathbf{i}} | g_{\sigma(i)} = g_i^{[1]} \;\text{ for all } i \in \mathbf{i}\}$$

with $GL_{a_{\mathbf{i}}}(\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}})$ by $g_{\mathbf{i}} := (g_i)_{i\in\mathbf{i}} \longmapsto g_{i_0}$ and identify

$$R_{\boldsymbol{\rho}}^F = \{(x_\rho)_{\rho\in\boldsymbol{\rho}} | x_{\sigma(\rho)} = x_\rho^{[1]} \;\text{ for all } \rho \in \boldsymbol{\rho}\}$$

with $\mathbb{F}_{q^{\varepsilon_{\boldsymbol{\rho}}}}^{a_{\mathbf{j}}\times a_{\mathbf{i}}}$ by $x_{\boldsymbol{\rho}} := (x_\rho)_{\rho\in\boldsymbol{\rho}} \longmapsto x_{\rho_0}$. Hence, we may identify their products

$$G(\beta)^F = \prod_{\mathbf{i}\in I} G_{\mathbf{i}}^F \;\text{ and }\; R(\beta)^F = \prod_{\boldsymbol{\rho}\in\Gamma_1} R_{\boldsymbol{\rho}}^F,$$

with

$$G := \prod_{\mathbf{i}\in I} GL_{a_{\mathbf{i}}}(\mathbb{F}_{q^{\varepsilon_{\mathbf{i}}}}) \;\text{ and }\; X := \prod_{\boldsymbol{\rho}\in\Gamma_1} \mathbb{F}_{q^{\varepsilon_{\boldsymbol{\rho}}}}^{a_{\mathbf{j}}\times a_{\mathbf{i}}}$$

respectively. Under this identification, the action of $G(\beta)^F$ on $R(\beta)^F$ becomes the action of $G$ on $X$: for $g = (g_{\mathbf{i}}) \in G$ and $x = (x_{\boldsymbol{\rho}}) \in X$,

$$(g \cdot x)_{\boldsymbol{\rho}} = g_{\mathbf{j}}^{[s_\rho]} x_{\boldsymbol{\rho}} (g_{\mathbf{i}}^{[r_\rho]})^{-1} \text{ for each arrow } \boldsymbol{\rho} : \mathbf{i} \to \mathbf{j},$$

where $0 \leqslant r_{\boldsymbol{\rho}} \leqslant \varepsilon_{\mathbf{i}} - 1$ and $0 \leqslant s_{\boldsymbol{\rho}} \leqslant \varepsilon_{\mathbf{j}} - 1$ are determined by $\rho_0 : \sigma^{r_\rho}(i_0) \to \sigma^{s_\rho}(j_0)$ (see Remark 6.7(a)). Then $M_{Q,\sigma}(\alpha, q)$ is the number of $G$-orbits in $X$.

For each $g = (g_{\mathbf{i}}) \in G$, we set $X^g = \{x \in X \mid g \cdot x = x\}$ and $G_g = \{h \in G \mid hg = gh\}$. By Burnside's formula, we have

$$M_{Q,\sigma}(\alpha, q) = \frac{1}{|G|}\sum_{g\in G} X^g = \sum_{g\in \operatorname{ccl}(G)} \frac{|X^g|}{|G_g|},$$

where $\mathrm{ccl}(G)$ is a set of representatives of conjugacy classes of $G$. Further, we have

$$|X^g| = \prod_{\boldsymbol{\rho}:\mathbf{i}\to\mathbf{j}} |X^g_{\boldsymbol{\rho}}| \ \text{ and } \ |G_g| = \prod_{\mathbf{i}\in I} |GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}})_{g_\mathbf{i}}|$$

where $X^g_{\boldsymbol{\rho}} = \{x_{\boldsymbol{\rho}} \in \mathbb{F}^{a_\mathbf{j}\times a_\mathbf{i}}_{q^{\varepsilon_{\boldsymbol{\rho}}}} \mid g_\mathbf{j}^{[s_\rho]} x_{\boldsymbol{\rho}} = x_{\boldsymbol{\rho}} g_\mathbf{i}^{[r_\rho]}\}$ and $GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}})_{g_\mathbf{i}} = \{h_\mathbf{i} \in GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}}) \mid h_\mathbf{i} g_\mathbf{i} = g_\mathbf{i} h_\mathbf{i}\}$.

In order to compute $M_{Q,\sigma}(\alpha, q)$, we need to deal with the conjugacy classes in $GL_m(\mathbb{F}_{q^r})$ for $m, r \geqslant 1$ (see, for example, [22, Chap.IV]).

We denote by $\Phi(q^r)$ the set all irreducible polynomials in $T$ over $\mathbb{F}_{q^r}$ with leading coefficients 1, excluding the polynomial $T$, and by $\mathcal{P}$ the set of all partitions, i.e. finite sequences $\lambda = (\lambda_1, \lambda_2, \cdots)$ of non-negative integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. For $\lambda, \mu \in \mathcal{P}$, we define $|\lambda| = \sum_i \lambda_i$, $b_\lambda(q) = \prod_{i\geqslant 1}(1-q)(1-q^2)\cdots(1-q^{\lambda_i})$, and $\langle \lambda, \mu\rangle = \sum_{i,j} \min\{\lambda_i, \mu_j\}$.

It is known that the conjugacy classes in $GL_m(\mathbb{F}_{q^r})$ are in one-to-one correspondence with the isoclasses of $\mathbb{F}_{q^r}[T]$-modules of dimension $m$. The latter is parametrized by the functions $\pi : \Phi(q^r) \to \mathcal{P}$ such that $\sum_{\varphi\in\Phi(q^r)} d(\varphi)|\pi(\varphi)| = m$, where $d(\varphi)$ denotes the degree of $\varphi$. Each such a function $\pi$ corresponds to the module

$$\bigoplus_{\varphi\in\Phi(q^r)} \bigoplus_{i\geqslant 1} \mathbb{F}_{q^r}[T]/(\varphi^{\pi_i(\varphi)}),$$

where $\pi(\varphi) = (\pi_1(\varphi), \pi_2(\varphi), \cdots)$.

Let $g \in GL_m(\mathbb{F}_{q^r})$ be in the conjugacy class corresponding to the partition function $\pi : \Phi(q^r) \to \mathcal{P}$. Then we have (see [22, p.272])

$$|GL_m(\mathbb{F}_{q^r})_g| = |\{h \in GL_m(\mathbb{F}_{q^r}) | hg = gh\}| = \prod_{\varphi\in\Phi(q^r)} q^{rd(\varphi)\langle\pi(\varphi),\, \pi(\varphi)\rangle} b_{\pi(\varphi)}(q^{-rd(\varphi)}).$$

Further, let $s, t, n_1, n_2, n \geqslant 1$ be such that $n_1|n$ and $n_2|n$. Take $g_1 \in GL_s(\mathbb{F}_{q^{n_1}})$ and $g_2 \in GL_t(\mathbb{F}_{q^{n_2}})$ such that their conjugacy classes correspond respectively to partition functions $\pi^1 : \Phi(q^{n_1}) \to \mathcal{P}$ and $\pi^2 : \Phi(q^{n_2}) \to \mathcal{P}$. By [16, p.253], we have

$$|\{x \in \mathbb{F}^{t\times s}_{q^n} | g_2 x = x g_1\}| = q^{n\sum_{\varphi\in\Phi(q^{n_1}),\psi\in\Phi(q^{n_2})} d(\varphi,\psi)\langle\pi^1(\varphi),\, \pi^2(\psi)\rangle},$$

where $d(\varphi, \psi)$ denotes the degree of the greatest common divisor of $\varphi$ and $\psi$ over a common extension field of $\mathbb{F}_{q^{n_1}}$ and $\mathbb{F}_{q^{n_2}}$ (Note that $d(\varphi, \psi)$ is independent of the extension field).

It follows that the conjugacy classes in $G = \prod_{\mathbf{i}\in I} GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}})$ are in one-to-one correspondence to multi-partition functions $\pi = (\pi^\mathbf{i})_{\mathbf{i}\in I}$ of $\pi^\mathbf{i} : \Phi(q^{\varepsilon_\mathbf{i}}) \to \mathcal{P}$ with $\sum_{\varphi\in\Phi(q^{\varepsilon_\mathbf{i}})} d(\varphi)\pi^\mathbf{i}(\varphi) = a_\mathbf{i}$. The set of all such multi-partition functions is denoted by $\mathfrak{P}$. For each $\pi \in \mathfrak{P}$, choose an element $g^\pi = (g^\pi_\mathbf{i})_\mathbf{i} \in G$ such that its conjugacy class corresponds to $\pi$. Thus, we have

$$|G_{g^\pi}| = \prod_{\mathbf{i}\in I} |GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}})_{g^\pi_\mathbf{i}}| = \prod_{\mathbf{i}\in I} \prod_{\varphi\in\Phi(q^{\varepsilon_\mathbf{i}})} q^{\varepsilon_\mathbf{i} d(\varphi)\langle\pi^\mathbf{i}(\varphi),\, \pi^\mathbf{i}(\varphi)\rangle} b_{\pi^\mathbf{i}(\varphi)}(q^{-\varepsilon_\mathbf{i} d(\varphi)}).$$

Clearly, for each $\mathbf{i} \in I$, $\pi^\mathbf{i}$ corresponds to the conjugacy class of $g^\pi_\mathbf{i}$ in $GL_{a_\mathbf{i}}(\mathbb{F}_{q^{\varepsilon_\mathbf{i}}})$. Let $0 \leqslant s \leqslant \varepsilon_\mathbf{i} - 1$ and denote by $\pi^\mathbf{i}[s]$ the partition function $\Phi(q^{\varepsilon_\mathbf{i}}) \to \mathcal{P}$ corresponding to the conjugacy class of $(g^\pi_\mathbf{i})^{[s]}$. Then we get

$$|X^{g^\pi}| = \prod_{\boldsymbol{\rho}:\mathbf{i}\to\mathbf{j}} |X^{g^\pi}_{\boldsymbol{\rho}}| = \prod_{\boldsymbol{\rho}:\mathbf{i}\to\mathbf{j}} q^{\varepsilon_{\boldsymbol{\rho}} \sum_{\varphi\in\Phi(q^{\varepsilon_\mathbf{i}}),\, \psi\in\Phi(q^{\varepsilon_\mathbf{j}})} d(\varphi,\psi)\langle\pi^\mathbf{i}[r_\rho](\varphi),\, \pi^\mathbf{j}[s_\rho](\psi)\rangle}.$$

Finally, we deduce

(9.0.2)

$$M_{Q,\sigma}(\alpha,q)=\sum_{\pi\in\mathfrak{P}}\frac{|X^{g^\pi}|}{|G_{g^\pi}|}=\sum_{\pi\in\mathfrak{P}}\frac{\prod_{\boldsymbol{\rho}:\mathbf{i}\to\mathbf{j}}q^{\varepsilon_{\boldsymbol{\rho}}\sum_{\varphi\in\Phi(q^{\varepsilon_{\mathbf{i}}}),\,\psi\in\Phi(q^{\varepsilon_{\mathbf{j}}})}d(\varphi,\psi)\langle\pi^{\mathbf{i}}[r_{\boldsymbol{\rho}}](\varphi),\,\pi^{\mathbf{j}}[s_{\boldsymbol{\rho}}](\psi)\rangle}}{\prod_{\mathbf{i}\in I}\prod_{\varphi\in\Phi(q^{\varepsilon_{\mathbf{i}}})}q^{\varepsilon_{\mathbf{i}}d(\varphi)\langle\pi^{\mathbf{i}}(\varphi),\,\pi^{\mathbf{i}}(\varphi)\rangle}b_{\pi^{\mathbf{i}}(\varphi)}(q^{-\varepsilon_{\mathbf{i}}d(\varphi)})}.$$

An orientation of the underlying graph $\overline{Q}$ of $Q$ is called $\sigma$-admissible[3] if it is compatible with the graph automorphism of $\overline{Q}$ induced by $\sigma$. Obviously, the orientation of $Q$ itself is $\sigma$-admissible.

**Theorem 9.1.** *The number $M_{Q,\sigma}(\alpha,q)$ is a polynomial in $q$ with rational coefficients and independent of the $\sigma$-admissible orientation of $Q$.*

*Proof.* By (9.0.2), $M_{Q,\sigma}(\alpha,q)$ is a rational function in $q$ over $\mathbb{Q}$. Since $M_{Q,\sigma}(\alpha,q)\in\mathbb{Z}$ for all power $q=p^s$ of a prime $p$, it follows that $M_{Q,\sigma}(\alpha,q)$ is a polynomial. Further, whenever $i_0\in\mathbf{i}$ and $\rho_0\in\boldsymbol{\rho}$ for $\mathbf{i}\in I$ and $\boldsymbol{\rho}\in\Gamma_1$ are fixed, the numbers $r_{\boldsymbol{\rho}}$ and $s_{\boldsymbol{\rho}}$ for each $\boldsymbol{\rho}\in\Gamma_1$ are clearly independent of the orientation of $\boldsymbol{\rho}$, we have again by (9.0.2) that $M_{Q,\sigma}(\alpha,q)$ is independent of the orientation of $\Gamma$, i.e., the $\sigma$-admissible orientation of $Q$. $\square$

By induction on the height $\mathrm{ht}\alpha:=\sum_{\mathbf{i}\in I}a_{\mathbf{i}}$ of $\alpha=\sum_{\mathbf{i}\in I}a_{\mathbf{i}}\mathbf{i}\in\Delta(\Gamma)^+$, we deduce the following (see [15] for the case of natural $\mathbb{F}_q$-modulated quivers).

**Corollary 9.2.** *The number $I_{Q,\sigma}(\alpha,q)$ of isoclasses of indecomposable $A^F$-modules of dimension vector $\alpha$ is a polynomial in $q$ with rational coefficients and independent of the $\sigma$-admissible orientation of $Q$.*

## 10. Roots and indecomposable $F$-stable representations

In this last section, we present an application to Lie theory. We keep the notation introduced in §9. Thus, $(Q,\sigma)$ denotes an ad-quiver and $\mathfrak{M}_{Q,\sigma}$ is the associated modulated quiver with the underlying valued quiver $\Gamma=\Gamma(Q,\sigma)=(I,\Gamma_1)$.

The quiver $Q$ defines a symmetric generalized Cartan matrix $C_Q=(a_{ij})_{i,j\in Q_0}$ given by

$$a_{ij}=\begin{cases}2 & \text{if } i=j\\ -|\{\text{arrows between } i \text{ and } j\}| & \text{if } i\neq j\end{cases}$$

while the valued quiver $\Gamma$ defines a symmetrizable generalized Cartan matrix $C_\Gamma=(b_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I}$ given by

$$b_{\mathbf{ij}}=\begin{cases}2 & \text{if } \mathbf{i}=\mathbf{j}\\ -\sum_{\boldsymbol{\rho}}\varepsilon_{\boldsymbol{\rho}}/\varepsilon_{\mathbf{i}} & \text{if } \mathbf{i}\neq\mathbf{j}\end{cases}$$

where the sum is taken over all arrows $\boldsymbol{\rho}$ between $\mathbf{i}$ and $\mathbf{j}$ (see (6.2.1)). In fact, all symmetrizable generalized Cartan matrix can be obtained in this way.

Let $\Delta(\Gamma)\subset\mathbb{Z}I$ be the root system associated with the valued quiver $\Gamma$, or equivalently, the root system of Kac-Moody algebra associated with the Cartan matrix $C_\Gamma$ (see [19] and [21] for definition). We shall write $\Delta(Q)$ for $\Delta(\Gamma)$ if $\sigma=1$ and $\Delta(\Gamma)^+$ for the positive subsystem. For $\beta\in\Delta(Q)^+$, let $t\geqslant 1$ be the minimal number satisfying $\sigma^t(\beta)=\beta$. We call $t$ the *$\sigma$-period* of $\beta$, denoted $p(\beta)=p_\sigma(\beta)$.

Recall from (9.0.1) the group isomorphism $\hat\sigma:(\mathbb{Z}Q_0)^\sigma\to\mathbb{Z}I$ which induces a map from $\Delta(Q)$ to $\Delta(\Gamma)$ in the following.

[3]The $\sigma$-admissible orientations of the underlying graph of $Q$ correspond to the orientations of the underlying graph of $\Gamma$.

**Lemma 10.1.** *(*[27],[17, Prop.4]*) Let $\beta \in \Delta(Q)$ and let*
$$\tilde{\beta} := \beta + \sigma(\beta) + \cdots + \sigma^{t-1}(\beta) \in (\mathbb{Z}Q_0)^\sigma,$$
*where $t = p(\beta)$. Then $\beta \longmapsto \hat{\sigma}(\tilde{\beta})$ induces a surjective map $\Delta(Q) \to \Delta(\Gamma)$. Moreover, if $\hat{\sigma}(\tilde{\beta})$ is real, then $\beta$ is real and is unique up to $\sigma$-orbit.*

Let $A$ be the path algebra of $Q$ and $F$ the Frobenius morphism induced from $\sigma$.

**Proposition 10.2.** *The dimension vector of each indecomposable $A^F$-module lies in $\Delta(\Gamma)^+$. Moreover, if $\alpha \in \Delta(\Gamma)^+$ is real, there is a unique, up to isomorphism, indecomposable $A^F$-module with dimension vector $\alpha$. In other words, we have*
(a) *If $I_{Q,\sigma}(q, \alpha) \neq 0$, then $\alpha \in \Delta(\Gamma)^+$.*
(b) *If $\alpha \in \Delta(\Gamma)^+$ is real, then $I_{Q,\sigma}(q, \alpha) = 1$.*

*Proof.* Let $X$ be an indecomposable $A^F$-module with $\mathrm{End}_{A^F}(X)/\mathrm{Rad}\,(\mathrm{End}_{A^F}(X)) \cong \mathbb{F}_{q^r}$. By Theorem 5.1, there is an indecomposable $A$-module $M$ such that $M = M^{[r]}$ and
$$X_k := X \otimes_{\mathbb{F}_q} k \cong M \oplus M^{[1]} \oplus \cdots M^{[r-1]}.$$
Moreover, $M, M^{[1]}, \cdots$, and $M^{[r-1]}$ are pairwise non-isomorphic. By Kac's theorem, the dimension vector $\beta := \mathbf{dim}\, M$ lies in $\Delta(Q)^+$. Let $t$ be minimal such that $\sigma^t(\beta) = \beta$. Since $\mathbf{dim}\, M^{[1]} = \sigma(\mathbf{dim}\, M)$, we have $t|r$ and
$$\mathbf{dim}\, X_k = \beta + \sigma(\beta) \cdots + \sigma^{r-1}(\beta) = \frac{r}{t}\tilde{\beta}.$$
If $\beta$ is an imaginary root, then $\hat{\sigma}(\tilde{\beta})$ is an imaginary root in $\Delta(\Gamma)^+$. We conclude that
$$\mathbf{dim}\, X = \hat{\sigma}(\mathbf{dim}\, X_k) = \frac{r}{t}\hat{\sigma}(\tilde{\beta})$$
is an imaginary root in $\Delta(\Gamma)^+$. If $\beta$ is real, then $r = t$. This implies that $\mathbf{dim}\, X = \hat{\sigma}(\tilde{\beta})$ is a real root in $\Delta(\Gamma)^+$.

Now let $\alpha = \hat{\sigma}(\tilde{\beta})$ be a real root. Then, by Lemma 10.1, $\beta$ is real. Again by Kac's theorem, there is a unique indecomposable $A$-module $M$ with dimension vector $\beta$. Since $\mathbf{dim}\, M^{[t]} = \sigma^t(\mathbf{dim}\, M) = \mathbf{dim}\, M$, where $t$ is minimal such that $\sigma^t(\beta) = \beta$, we have $M^{[t]} \cong M$. Hence, by Theorem 5.1, we obtain an $F$-stable module
$$\tilde{M} \cong M \oplus M^{[1]} \oplus \cdots \oplus M^{[t-1]}$$
such that $\tilde{M}^F$ is indecomposable $A^F$-module whose dimension vector is $\hat{\sigma}(\tilde{\beta}) = \alpha$. The uniqueness of such an indecomposable module follows from the fact that $\beta$ is unique up to $\sigma$-orbit. □

**Remarks 10.3.** (1) The statements (a) and (b) in 10.2 together with

(c) If $\alpha \in \Delta(\Gamma)^+$ is imaginary, then the *polynomial* $I_{Q,\sigma}(q, \alpha) \neq 0$.

constituent the so-called Kac's Theorem (see [19]) when $\sigma = 1$. The proposition partially generalizes Kac's theorem for quivers (over a finite field) to a result for modulated quivers defined by $(Q, \sigma)$ with $\sigma \neq 1$. The generalization of Kac's Theorem was first formulated by Hua in [15, Thm 4.1]. Hubery in [17] provides a proof by using a classification of the so-called ii-indecomposable representations of an ad-quiver. However, both [15] and [17] treat *only* natural $\mathbb{F}_q$-modulated quivers (see Remark 6.7).

(2) By using Ringel-Hall algebra approach, it has been proved in [4] that, for any prime power $q$ and dimension vector $\alpha$, the *number* $I_{Q,\sigma}(\alpha, q) \neq 0$ if and only if $\alpha \in \Delta(\Gamma)^+$. This is stronger than the statement (c) for imaginary roots. It should be still interesting to find a direct proof for (c) in the modulated quiver case.

We end the paper with the following conjecture which provides a direct link of Kac's theorem with its generalization.

**Conjecture 10.4.** *Let $A$ be the path algebra of an ad-quiver $(Q, \sigma)$ without oriented cycles. Let $\beta \in \Delta(Q)^+$. Then there exists an indecomposable $A$-module $M$ with dimension vector $\beta$ such that $p_F(M) = p_\sigma(\beta)$.*

We claim that this conjecture implies 10.3(c) immediately. Indeed, for any $\alpha \in \Delta(\Gamma)^+$, there exists a $\beta \in \Delta(Q)^+$ such that $\alpha = \hat{\sigma}(\tilde{\beta})$. By Kac's theorem, we have $I_{Q,1}(q, \beta) \neq 0$ for some $q$. Now the existence of such an $M$ guarantees that the associated $F$-stbale module $\tilde{M}$ defined in (5.0.1) has dimension vector $\tilde{\beta}$. Thus, the indecomposable $\tilde{M}^F$ has dimension vector $\alpha$. Therefore, $I_{Q,\sigma}(q, \alpha) \neq 0$.

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